MANUAL

OF

TRIGONOMETRY.

FOR THE USE

OF

HIGH SCHOOLS, ACADEMIES AND COLLEGES.

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IN THE

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PREFACE.

In the following pages the author has endeavored to present a brief, but thorough and complete, course in Trigonometry. While non-essentials have been excluded, considerable attention has been given to some matters not discussed in any of the usual text-books. It is discouraging even to a bright student, to meet, in his work in the Integral Calculus, difficulties that should have been surmounted in his Trigonometrical course. The chapter on Algebraic Trigonometry will, it is hoped, lighten the labor of the student preparing for advanced work and broaden the views of the reader whose Trigonometry is merely a minor part of a symmetrical course of liberal study. The text, taken in connection with the exercises, (all of which should be worked out), will afford ample preparation for any University Mathematical course.

The author desires to express his sense of obligation to Prof. Irving Stringham for encouraging and valuable hints and suggestions.

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February, 1888.

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TRIGONOMETRY.

CHAPTER I.

1. Trigonometry is a branch of Mathematics, in which are discussed the properties, relations, and applications of certain functions of *angles.* These functions, to be hereafter defined, are called the Trigonometric functions, and are practically useful mainly in the solution of triangles.

Fig. 1.

2. An angle, AOB (Fig. 1), is considered as generated by the revolution of a line about the vertex, O, from the position OA to the position OB. O is called the *Angular Centre;* OA, the *Initial,* and OB, the *Terminal* line. We call the revolution, and the angle generated, *positive,* when the motion is opposite in direction to that of a watch-hand; *negative,* when the motion is in the direction of that of a watch-hand. Angles are assumed to be positive unless they are explicitly stated to be negative. Angular magnitude, thus considered, evidently admits of infinite extension; AOD may be generated by a simple passage of the moving line from OA to OD, or by a movement in which this line reaches OD after passing around O any number of times. This requires, of course, that the magnitude of the measuring arcs be similarly considered; hence the occurrence of angles of 540°, 630°, etc. (*i. e.*, the

angle as ordinarily taken, + any multiple of 360°). The plane space about O is, for convenience, divided into four equal parts, called *quadrants*, by two lines, viz., the *Initial* line, assumed horizontal and *unlimited in length*, and a perpendicular thereto through O. These quadrants are numbered as in the figure, and an angle lies in that quadrant in which is found its terminal line.

3. An angle is usually denoted by a single symbol, thus: x, y, A. In general this symbol will denote the *radial* measure of the angle, except when we deal with triangles. The radial measure (Ven. IV, 19) of an angle is the arc intercepted (by the initial and terminal lines) divided by the radius with which it is described from the vertex as a centre. The unit is the angle whose sides intercept on the circumference an arc equal in length to the radius, for, in the case of this angle, arc: radius = 1. As $2\pi R = 360°$, $R = 360 \div 2\pi = 180° \div 3.1415926 = 57°.29578$, or about 57°.3 (57° 17′ 45″, nearly). To pass from the "*degree*" measure of an angle to its "*radial*" measure, we therefore divide by 57°.3, and conversely.

The *complement* of an angle (or arc) is a right angle (or 90°), minus the angle (or arc); the *supplement* is the difference between the angle (or arc) and two right angles (180°).

4. Angles are dealt with by means of certain *Trigonometric Functions*. If a perpendicular be dropped from any point of the *Terminal* line on the (unlimited) Initial line, a right triangle is formed called the *Triangle of Reference*. The following ratios, involving its sides, are the *Trigonometric Functions* of the angle.

(1.) The perpendicular (p) divided by the hypothenuse, is the SINE (*sin*). Thus (Fig. 1), BM : OB = sin AOB.

(2.) The base (b) divided by the hypothenuse, is the CO-SINE (*cos*). Thus, cos AOB = OM : OB; cos AOD = OL : OD.

(3.) The perpendicular (p) divided by the base (b) is the TANGENT (*tan*). Thus, tan AOB = BM : OM.

(4.) The base (b) divided by the perpendicular (p) is the CO-TANGENT (*cot*). Thus, cot AOC = OK : KC.

(5.) The reciprocal of the sine is called the CO-SECANT (*csc*). Thus, csc AOB = OB : BM.

(6.) The reciprocal of the cosine is called the SECANT (*sec*). Thus, sec AOD = OD : OL.

Unity minus the cosine is called the VERSED-SINE (*versin*); thus versin AOB = 1 – OM : OB.

Unity minus the sine is called the COVERSED-SINE (*coversin*); thus coversin AOC = 1 – CK : CO

It will be noticed:—

(1.) That each ratio can have but one value for any one angle. For if the perpendicular were dropped from any other point of OB than B (*e. g.*, B′, C′, D′), the new triangles of reference would be similar to OBM, and hence the ratios of their sides would equal the corresponding ratios in OBM.

(2.) Neither the sine nor the cosine can exceed 1.

(3.) The secant and cosecant must both exceed 1.

(4.) Neither the versed-sine nor the coversed-sine can exceed 2.

5. The functions (**4**) may evidently be grouped in two triads, viz.: Sine, Tangent, Secant, and Co-sine, Co-tangent, Co-secant. Fig. 2 will render clear the reason of this nomenclature. Let AOB be any angle, BOQ its complement, BOM and KOL the triangles of reference. As BOM is similar to KOL,

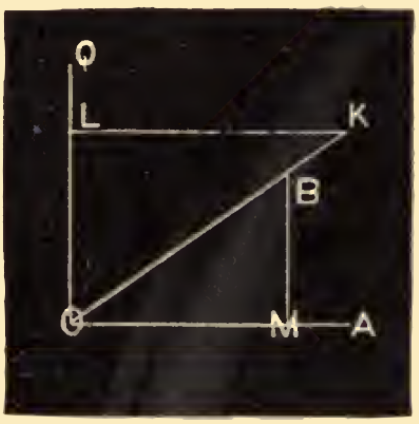


Fig. 2.

OM : OB = KL : OK, or co-sine AOB = sin (90° – AOB),

OM : BM = KL : OL, or co-tangent AOB = tan (90° – AOB),

OB : BM = OK : OL, or co-secant AOB = sec (90° – AOB),

that is, the *co*-sine of an angle is the sine of the *complement*, the *co*-tangent is the tangent of the *complement*, and the *co*-secant is the secant of the *complement*.

6. While each function has but one value for any one angle, the converse is not true. To a single numerical value of a function may correspond four different angles, each less than 360°. Any confusion that might result from this fact is avoided by the adoption of the following convention (Cf. Alg. §§ 8, 46, 55, 429, 432):—

Vertical distances are estimated *from* the *initial* line XOX′ (Fig. 1); those measured *upward* are said to be *positive;* hence those measured *downward* must be taken as *negative.* Horizontal distances are estimated *from* the vertical line YOY′; those laid off to the *right* are said to be *positive;* hence those measured to the *left* are *negative.* The hypothenuses of the triangles of reference being, in general, neither vertical nor horizontal, may be considered signless.

Consequently all functions of angles in the first quadrant are positive. For angles in the second quadrant the base of the triangles of reference are negative, and the perpendiculars positive; hence the sine and cosecant are *positive,* and the other functions *negative.* In the third quadrant both base and perpendicular are negative; hence the tangent and cotangent are positive, and the other functions negative. Finally, for angles in the fourth quadrant the cosine and secant are positive, and the other functions negative.

7. Limiting Values of Functions. *The Sine.* At 0° p in the triangle of reference is evidently 0, and hence $\sin 0° = 0 : h = 0$. It will be convenient, though not necessary, to assume that the length of h remains constant as the angle increases from 0° to 360°; let the changing angle be denoted by x. As x increases from 0° onward, p increases, and hence $\sin x$ increases, rapidly at first, but slowly as x approaches 90°. When x is nearly 90°, p and h are nearly equal, and, finally, when $x = 90°$, p and h coincide, and $p : h = 1$, $\therefore \sin 90° = 1$. As x increases, p (and hence $p : h$) decreases, slowly at first, but

more rapidly as x approaches 180°. When x is very near 180°, p (and hence $p : h$) is very small; finally when $x = 180°$, $p : h = 0$. A peculiar circumstance here presents itself. As $180° = 180° - 0°$, and also $180° + 0°$ (Alg. Ch. VIII) the angle of 180° may be considered as ending in either the second quadrant or the third. In the first case sin 180° is +, and in the second case −. Obliged to recognize both contingencies we say that sin $180° = \pm 0$. Here, then, sin x changes its sign, becoming negative. As x increases, p, and therefore $p : h$, increases numerically until, when $x = 270°$, p and h coincide, and $p : h = -1$. As x increases, p (and hence $p : h$) decreases numerically, until, when $x = 360°$, sin $x = \mp 0$ according as x is taken as $360° - 0°$ or $360° + 0°$. If the angle generatrix continue to revolve, the same values will evidently recur in the same order.

The Tangent. When $x = 0°$, $p = 0$, and $\therefore p : b = 0$, *i. e.*, tan 0° $= 0$. As x increases p *increases* and b *decreases at the same time;* hence tan x increases very rapidly. When x is nearly 90°, tan x is very great, and, when $x = 90°$, tan $x = p : 0 = \infty$. As x may at this stage be considered as ending in the first $(90° - 0°)$, or in the second $(90° + 0°)$, quadrant, and as the tangent is positive in the first and negative in the second, we say that tan $90° = \pm\infty$. Here the tangent changes sign by passing through ∞. As x increases toward 180° tan x rapidly diminishes (numerically), since p diminishes and b increases, until, when $x = 180°$, tan $x = \mp 0$. Tan x, changing its sign by passing through 0, increases rapidly as x increases toward 270°. When $x = 270°$, tan $x = \pm\infty$, and as x increases, tan x, changing its sign by passing through ∞, decreases, numerically, with great rapidity, until, when $x = 360°$ tan $x = \mp 0$.

A consideration of the changes in all the functions as x changes from 0° to 360° leads to the results given in the following

Table of Limiting Values of the Functions.

FUNCTION.	0° to 90°	90° to 180°	180° to 270°	270° to 360°
SINE.....	0 to +1	+1 to ±0	±0 to −1	−1 to ∓0
COSINE.....	+1 to ±0	±0 to −1	−1 to ∓0	∓0 to +1
TANGENT.....	0 to ±∞	±∞ to ∓0	∓0 to ±∞	±∞ to ∓0
COTANGENT.....	+∞ to ±0	±0 to ∓∞	∓∞ to ±0	±0 to ∓∞
SECANT.....	1 to ±∞	±∞ to −1	−1 to ∓∞	∓∞ to +1
COSECANT.....	∞ to +1	+1 to ±∞	±∞ to −1	−1 to ∓∞

8. Negative Angles. By comparing the functions of negative angles (**2**) with the functions of numerically equal positive angles, it will be seen that, A being any angle, $\sin(-A) = -\sin A$; $\csc(-A) = -\csc A$; $\cos(-A) = \cos A$; $\sec(-A) = \sec A$; $\tan(-A) = -\tan A$; $\cot(-A) = -\cot A$. Hence *changing the sign of an angle changes the sign of the* SINE, COSECANT, TANGENT, *and* COTANGENT, *but does not affect the sign of the* COSINE *and* SECANT.

9. Fundamental Relations.* If A represent any angle,

(1.) $\operatorname{Sin}^2 A + \cos^2 A = 1$.

(2.) $\operatorname{Tan} A = \dfrac{\sin A}{\cos A}$.

(3.) $\operatorname{Cot} A = \dfrac{\cos A}{\sin A}$.

(4.) $\operatorname{Tan} A \cot A = 1$.

(5.) $\operatorname{Sin} A \csc A = 1$.

(6.) $\operatorname{Cos} A \sec A = 1$.

(7.) $\operatorname{Sec}^2 A = 1 + \tan^2 A$.

(8.) $\operatorname{Csc}^2 A = 1 + \cot^2 A$.

Of these, (2), (3), (4), (5), (6), follow immediately from the definitions. To prove (1): $\operatorname{Sin} A = \dfrac{p}{h}$; $\cos A = \dfrac{b}{h}$ hence $\sin^2 A + \cos^2 A = \dfrac{p^2}{h^2} + \dfrac{b^2}{h^2} = \dfrac{p^2 + b^2}{h^2} = \dfrac{h^2}{h^2} = 1$. (7) and (8) may be similarly established.

10. Anti-functions. If $ab = m$, we write $b = a^{-1}m$. By analogy we usually say that if $\tan A = m$, $A = \tan^{-1} m$, al-

*The student must memorize these, and be thoroughly familiar with them.

though the *word* tangent cannot be considered a factor. Though this notation can be justified, it is inconvenient and may produce confusion, as powers of functions are indicated by placing the exponent above and a little to the right of the abbreviation for the function; thus $(\tan A)^3$ is usually written $\tan^3 A$; so that $\tan^{-1} A$ might mean "$\frac{1}{\tan A}$" as well as "the angle whose tangent is A." As it is useless now to attempt to change the notation, the student must be on his guard to avoid confusion. While $\sin^{-1} A$, $\cot^{-1} A$, etc., are sometimes read "anti-sine of *A*," "inverse sine of *A*," etc., it is better to read, in full, "the angle whose sine is A," etc.

11. The trigonometric functions may be represented by lines, if it be assumed that all angles are formed about the centre of a circle of unit radius, and that in every triangle of reference one side is equal to the radius. In this case, however, the angles are treated by means of the measuring arcs, and it is better to consider the functions as functions immediately of the arcs. Arcs considered directly are generally supposed to begin at the right hand extremity of a horizontal diameter (and we shall assume that they do so begin); this point is called the *primary origin*, or simply, the *origin*. The complements of arcs are in this case estimated from the upper extremity of a vertical diameter; this point is called the *secondary origin*.

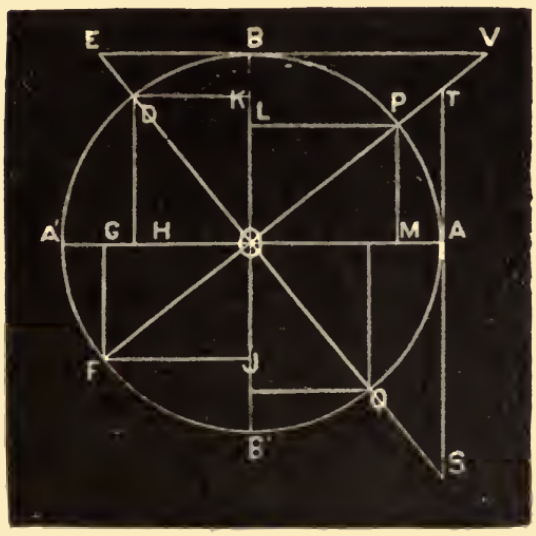


Fig. 3.

The *sine* is the perpendicular dropped from the termination of the arc on the diameter through the origin. Thus sin AP (Fig. 3) is PM. The *cosine* of an arc is the sine of the complement; thus cos AP = PL; cos AD = KD. The *tangent* is that part of the geometric tangent at the origin which is limited by the diameter through the termination of

the arc; thus tan AP = tan ABF = AT. The *cotangent* of an arc is the tangent of the complement; thus cot AD = BE. The *secant* is the line connecting the centre of the circle with the extremity of the tangent; thus sec ABD = OS. The *cosecant* of an arc is the secant of the complement; thus csc AP = OV. The *versed-sine* is the radius minus the cosine, and the *co-versed-sine* is the radius minus the sine; thus versin AP = MA; coversin AD = KB.

Lines are always measured *from* AA′ and BB′; those measured upward and those measured to the right of BB′ are positive; all other horizontal or vertical lines are negative. The secant and cosecant follow in sign their reciprocals, the cosine and sine, respectively.*

If the radius be different from unity the lines indicated above must be divided by the radius, in order to get the various functions. As these definitions agree in substance with those given in (4) we may use angles or their measuring arcs indifferently, and likewise the trigonometric functions or their line representatives.

Fig. 4.

12. *Functions of 30° and of 45°.* Let AOB (Fig. 4) = 30°. Then (Ven. 36 p. 51) OB = 2 AB, ∴ sin AOB = AB : OB = $\frac{1}{2}$. As $\sin^2 30° + \cos^2 30° = 1$, $\cos^2 30° = \frac{3}{4}$ ∴ cos 30° = $\frac{1}{2}\sqrt{3}$. Tan 30° = $\frac{1}{2} \div \frac{1}{2}\sqrt{3} = \frac{1}{3}\sqrt{3}$; cot 30° = $\sqrt{3}$; sec 30° = $\frac{2}{3}\sqrt{3}$; csc 30° = 2.

Fig. 5.

Again let AOC (Fig. 5) = 45°. As OC is the diagonal of a square, AC : OC = sin 45° = $\frac{1}{2}\sqrt{2}$ = cos 45°. As OA = AC, tan 45° = cot 45° = 1. Finally, sec 45° = csc 45° = $\sqrt{2}$.

13. Trigonometric Tables.—To enable us to deal with angles by means of their functions tables are computed, giving the values of these functions for all angles from 0° to 90°.

*The versed-sine and coversed-sine are measured *from* the foot of the sine and cosine respectively, and therefore are never negative.

The angles are usually taken at intervals of $1'$, though the interval may be as large as $10'$ in tables for ordinary use, and is frequently as small as $1''$ in tables used for calculations requiring great accuracy. The functions of an angle lying between any two angles of the table are found by interpolation (Alg. 491 and Appendix III).

14. The Tables of **Natural Functions** give the actual values of the functions; the Tables of **Logarithmic Functions** give the common logarithms of these values. As *all* sines and cosines, tangents of angles *between 0° and 45°*, and cotangents of angles between 45° and 90°, are "proper" fractions, the characteristics of their logarithms are negative. To avoid these negative characteristics, *10 is added to the characteristic of the logarithm* of each of these functions, and, in many tables, for uniformity, 10 is added to *every* characteristic.

15. Elementary Method of Computing a Table of Natural Functions.—We will consider the line representatives of functions, *i. e.*, (**11**) the functions of the *arc* (radius being unity). Though an arc exceeds its sine, the inequality diminishes as the arc decreases, the ratio $(\sin x) : x$ having for its limit 1. This inequality is so small for an arc of $1'$ that there is no error, within the limits of ordinary tables, in assuming arc $1' = \sin 1'$. In a circle of radius 1, the semicircumference $= 10800' = \pi = 3.14159265$; hence arc $1' = 3.14159265 \div 10800 = .000290888 = \sin 1'$. $\cos 1' = \sqrt{1 - \sin^2 1'}$

$$= \sqrt{(1 + \sin 1')(1 - \sin 1')} = \sqrt{1.000290888 \times .999709112}$$

$$= .999999958$$

It will be shown (**37**) that if A and B be any two angles,

(I) $\sin (A + B) = 2 \sin A \cos B - \sin (A - B)$.

(II) $\cos (A + B) = 2 \cos A \cos B - \cos (A - B)$.

Let $B = 1'$, and let A become successively, $1'$, $2'$, $3'$, etc.

From (I), $\sin (1' + 1') = \sin 2' = 2 \sin 1' \cos 1'$

Sin $(2' + 1') = \sin 3' = 2 \sin 2' \cos 1' - \sin 1'$

$\sin 4' = 2 \sin 3' \cos 1' - \sin 2'$,

and so on.

From (II) $\cos 2' = 2\cos^2 1' - 1$

$$\cos 3' = 2\cos 2'\cos 1' - \cos 1'$$
$$\cos 4' = 2\cos 3'\cos 1' - \cos 2'$$

and so on.

As $2\cos 1' = 1.999999154 = 2 - .000000846$ is a constant factor in the first term of the second member, these equations may be written

$$\sin 2' = 2\sin 1' - .000000846\sin 1' \qquad = .000581776$$
$$\sin 3' = 2\sin 2' - .000000846\sin 2' - \sin 1' = .000872665$$
$$\sin 4' = 2\sin 3' - .000000846\sin 3' - \sin 2' = .001163553$$

and similarly for the cosines. After computing the functions of angles up to 30° by these formulæ, we let A (I) remain equal to 30°, while B changes from 1′ successively at minute intervals up to 15°. Then, as $\sin 30° = \frac{1}{2}$, $2\sin 30° = 1$, and we have

$$\sin(30° + 1') = \cos 1' - \sin 29°\ 59'$$
$$\sin(30° + 2') = \cos 2' - \sin 29°\ 58'$$
$$\sin(30° + 3') = \cos 3' - \sin 29°\ 57'$$

and so on up to 45°, the only operation required in the second members being mere subtraction.

Instead of (II) we now use, (**37**), the formula

$$\cos(A + B) = -2\sin A\sin B + \cos(A - B).$$

Letting A remain equal to 30°, we have

$$\cos(30° + 1') = -\sin 1' + \cos 29°\ 59'$$
$$\cos(30° + 2') = -\sin 2' + \cos 29°\ 58'$$

and so on, up to 45° inclusive.

We need not compute beyond 45°, as the sine and cosine of any angle between 45° and 90° are respectively the cosine and sine of the complement, *i. e.*, of an angle between 45° and 0° (of which the cosine and sine have been computed).

The tangents, cotangents, secants, and cosecants may be obtained from the sines and cosines by means of the fundamental relations (**9**). Functions of angles greater than 90° may be obtained from the table by means of the relations suggested in (**5, 6**).

16. The Table of Logarithmic Functions may be obtained by simply tabulating the common logarithms of the natural functions and adding 10 to each characteristic. Other and better methods will be indicated later. This table is usually employed in preference to the table of natural functions, as logarithms are utilized whenever they serve to facilitate computation, as in multiplications, divisions, involution, and evolution.

17. Tables are usually so arranged that for angles from 0° to 45° the degrees and the names of the functions are to be taken from the *top* of the page, and the minutes from the left-hand column, while for angles between 45° and 90° the degrees and the names of the functions are to be taken from the *bottom* of the page, and the minutes from the right-hand column.

18. Interpolation is effected on the hypothesis that as the arc changes uniformly, the function changes uniformly. The error involved in this assumption being greatest for *cosines of very small angles* and *sines of angles near 90°*, we avoid computing small angles by means of the cosine, and angles near 90° by means of the sine, unless we have tables specially calculated for the purpose. It is best in general to compute angles by means of the tangent or cotangent.

19. To guard against the errors likely to creep into the extended computations of (**15**) we check the results of those computations by comparing them with the values of functions of various angles, determined independently by formulæ that might in this connection be termed verification formulæ. Numerous examples of these will be given later.

CHAPTER II.

SOLUTION OF TRIANGLES.

The *parts* of a triangle, trigonometrically considered, are its sides and angles. Plane Trigonometry treats in detail the problem:—"Given three parts of a triangle, to compute the other three," or, more generally, "From data sufficient to determine a plane triangle, to compute the unknown parts." Of given parts one at least must fix the length of a line; angles only can determine nothing more than the *shape* of a triangle:—the *ratios* of the sides to each other.

In any triangle, ABC, the angles will be denoted by A, B, C, and the sides opposite by a, b, c, respectively.

20. Right Triangles.—The problems that arise in the solution of right triangles may be grouped thus:—

I. Given the hypothenuse and another side.
II. Given the two sides about the right angle.
III. Given the hypothenuse and an acute angle.
IV. Given an acute angle and a side, opposite or adjacent.

The definitions (4) may, (Fig. 6) be stated thus:—

(1) $\sin C = c \div a$
(2) $\cos C = b \div a$
(3) $\tan C = c \div b$
(4) $\cot C = b \div c$

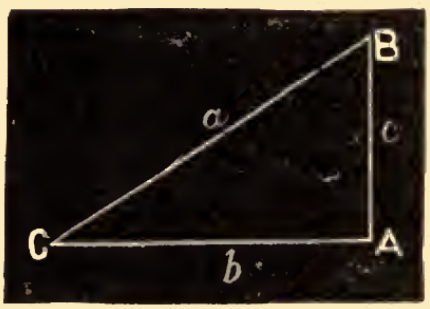


Fig. 6.

These suffice for the solution of all right triangles.

I. 1°. Given a, b. (2) gives cos C, whence C is obtained at once from the tables. $c^2 = a^2 - b^2$; hence $c = \sqrt{(a+b)\ (a-b)}$ and the solution is complete. c might also be computed from (1) after C has been found; but, (1) any error in the computation of C would vitiate this computation, and (2) if C be near 90° its sine cannot be accurately determined from the usual

tables. Required parts should always be determined directly (if possible) from given parts. Unused relations may then serve as "checks" on the work.

2°. Given a, c. Solution similar to that just indicated.

II. Given b, c. From (3) and the Tables, C and hence B are determined at once. To determine a we have the two formulæ $a = \sqrt{b^2 + c^2}$ and $a = c \div \sin C$; the former is direct, but not adapted to logarithmic computation; the latter, the simpler, has the disadvantage already indicated.

III. Given b, B. $C = 90° - B$. (2) gives a, and (3) determines c. When c, C are given the solution is similar.

21. Oblique Triangles.—As every oblique triangle may be divided into two right triangles, all oblique triangles may be solved by (**20**).

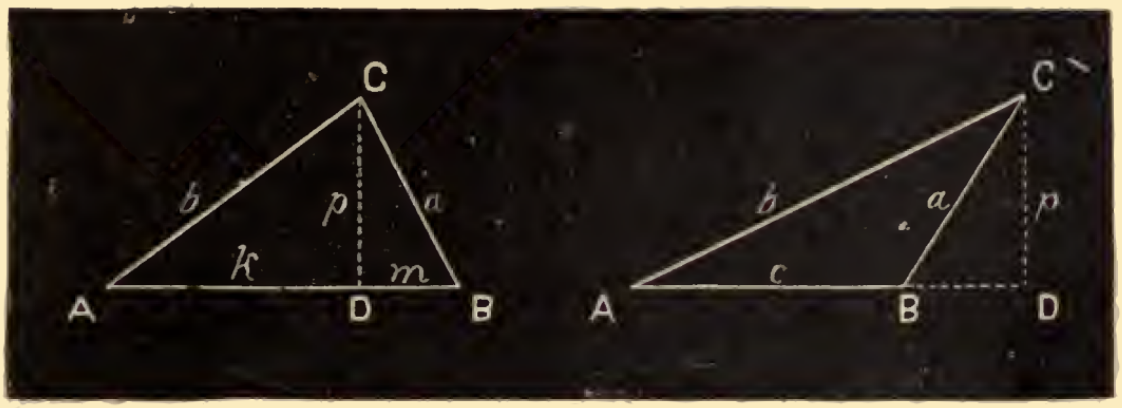


Fig. 7.

Fig. 8.

I. *Given two sides and the included angle*, e. g., c, b, A. Draw from the extremity of one known side, (b), CD $\perp c$, the other known side. Solve ACD (**20**, III), thus determining k and p; if $k > c$, CD falls without the triangle (Fig. 7). $BD = c - k$ or $k - c$, according as CD falls within or without. Knowing p and m, solve the triangle BDC, thus obtaining a and BCD. $ACB = ACD + BCD$ or $ACD - BCD$ according as CD falls within or without ACB.

II. *Given two angles and a side*, e. g., A, B, b. All the angles are known, and the side b. Draw, from the extremity of this side, CD $\perp c$. Solve ACD (I). Then, knowing p and the angle DCB, solve DCB. If CD falls within the tri-

angle, *i. e.*, if ACD $<$ C (as given), AB = AD + DB. If CD falls without, *i. e.*, if ACD $>$ C, AB = AD − DB.

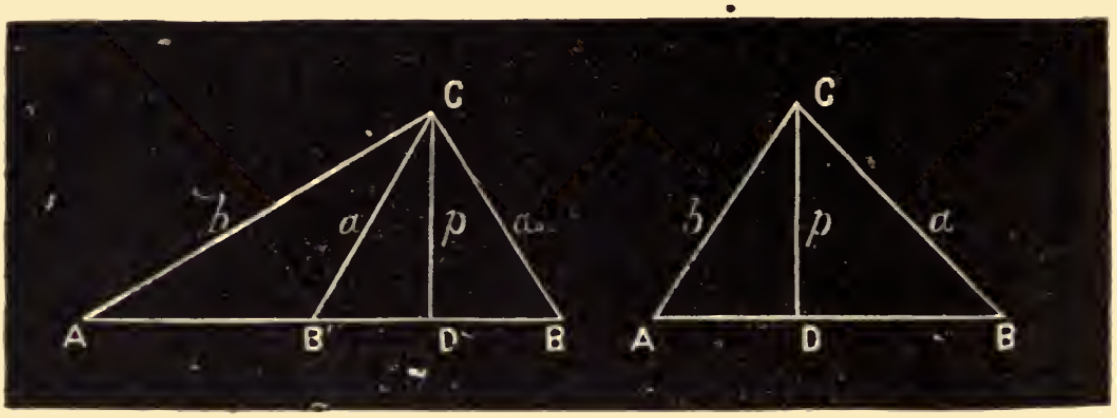


Fig. 9. Fig. 10.

III. *Given two sides and the angle opposite one of them,* e. g., *b*, *a* *A*. From the extremity of a known side, *b*, draw CD $\perp$ *c*, the unknown side. Solve ACD, as before. Knowing *p* and *a*, solve DCB (**20**, I). As in this case two triangles may fulfill the given conditions, it is called the "*ambiguous case.*" Thus if $a < b$, *a* may take either one of the positions CB or CB', cutting *c* at equal distances from the foot of the perpendicular, and both ACB and ACB' fulfill the requirements. If $b = a$, ACB' shrinks to a line, and there is but one solution. Again if $a > b$, *a* can fall on only one side of AC and there can be but one solution. If $a = p$, ACB' and ACB coincide. Finally, if $a < p$ there is no solution.

IV. *Given the sides a, b, c.* Drop the perpendicular from any vertex, as C. Then (Fig. 7),

$$p^2 = b^2 - k^2 = a^2 - m^2$$

$$\therefore b^2 - a^2 = k^2 - m^2, \text{ whence}$$

$$(b-a)(b+a) = (k-m)(k+m), \text{ i. e.,}$$

$$b+a : k+m :: k-m : b-a \quad (1);$$

$$\text{or } k - m = \frac{(b+a)(b-a)}{c} \quad (2).$$

Determining $k - m$ from (2), we have

$$k = \tfrac{1}{2}(k+m) + \tfrac{1}{2}(k-m), \text{ and}$$

$$m = \tfrac{1}{2}(k+m) - \tfrac{1}{2}(k-m).$$

Knowing k and m, we solve ACD and DCB, and thus determine A, B, C.

22. The solutions, I, III, (**21**) may be abbreviated by noting that "*the sides of a plane triangle are proportional to the sines of the opposite angles.*" Thus (Figs. 9, 10), $p = b \sin A = a \sin B$, whence $a : b = \sin A : \sin B$.

This proposition may be used to "check" solutions of (**21**).

The constant ratio $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ is the *modulus* of the triangle, and is numerically equal to the diameter of the circumscribing circle.

NUMERICAL EXAMPLES.

I. Right Triangles.—1. Let $a = 256$, $b = 172$. From (2) $\cos C = \sin B = 172 \div 256 = .67188$. The table of Natural Functions gives $C = 47° 47' 14''$, and hence $B = 42° 12' 46''$. $c = \sqrt{(a+b)(a-b)} = \sqrt{428 \times 84} = \sqrt{35952} = 189.6$. (1) now serves as a "check," to test the correctness of the computation. Thus $\sin 47° 47' 14'' = .74065$, $\therefore c = 256 \times .74065 = 189.6$ as before.

2. Let $b = .0795$; $c = .0386$ [Logarithmic solution]. From (3), $\tan C = c \div b$, whence $\log \tan C = \log c - \log b + 10 = \bar{2}.58659 - \bar{2}.90037 + 10 = 9.68622$, whence $C = 25° 53' 26''$. From (1), $a = c \div \sin C$, $\therefore \log a = \log c - \log \sin C + 10 = \bar{2}.58659 - 9.64025 + 10 = \bar{2}.94634$ $\therefore a = 0.08838$. The relation $a^2 = b^2 + c^2$ may be used as a "check."

3. Let $a = 27.125$, $C = 37° 42' 18''$. From (1), $c = a \sin C$

$\log a = 1.43337$
$\log \sin C = 9.78647$
$\log c = 1.21984$
$c = 16.589$

Again, $b = a \sin B$;

$\log a = 1.43337$
$\log \sin B = 9.89827$
$\log b = 1.33164$
$b = 21.46$

II. Oblique Triangles.—1°. Given $A = 56° 41'$, $B = 74° 28'$

and $b = 1326$. $C = 180° - (A + B) = 48° 51'$. In ACD (Fig. 7) $AD = b \cos A$ and $CD = b \sin A$.

$$\log 1326 = 3.12254 \qquad \log 1326 = 3.12254$$
$$\log \cos 56° 41' = 9.73978 \qquad \log \sin 56° 41' = 9.92202$$
$$\log AD = 2.86232 \qquad \log CD = 3.04456$$
$$\therefore AD = 728.31$$

In CDB, $CB = CD \div \sin B$, and $DB = CD.\cot B$.

$$\log CD = 3.04456 \qquad \log CD = 3.04456$$
$$\log \sin 74° 28' = 9.98384 \qquad \log \cot 74° 28' = 9.44397$$
$$\log CB = 3.02840 \qquad \log DB = 2.48853$$
$$a = 1067.59 \qquad DB = 307.99$$
$$c = AD + DB = 1036.3$$

We check our results by the law of sines; thus: $\sin A \div a$ should equal $\sin B \div b$.

$$\log \sin B = 9.98384 \qquad \log \sin A = 9.87679$$
$$\log b = 3.12254 \qquad \log a = 3.01549$$
$$6.86130 \qquad 6.86130$$

Thus the logarithms of the two ratios agree exactly.

2. Given $a = 1200$, $b = 758$, $A = 72° 14'$. As $a > b$ there is but one solution.

In ACD, $CD = b \sin A$, and $AD = b \cos A$.

$$\log \sin 72° 14' = 9.97878 \qquad \log \cos 72° 14' = 9.48450$$
$$\log 758 = 2.87967 \qquad = 2.87967$$
$$\log CD = 2.85845 \qquad \log AD = 2.36417$$
$$CD = 721.8 \qquad AD = 231.3$$

In CDB, $BD = \sqrt{(a + CD)(a - CD)} = \sqrt{1921.8 \times 478.2}$

$$\log 1921.8 = 3.28371 \qquad \log \sin B = \log CD - \log a + 10$$
$$\log 478.2 = 2.67961 \qquad \log CD = 2.85845$$
$$2 \mid 5.96332 \qquad \log a = 3.07918$$
$$\log BD = 2.98166 \qquad \log \sin B = 9.77927$$
$$BD = 958.6 \qquad B = 36° 58' 49''$$

$ACB = 180° - (A + B) = 70° 47' 11''$.

3. Given $a = 12.76$; $b = 26.58$, $c = 31.6$. Then, (Fig. 7),

$k - m = \dfrac{39.34 \times 13.82}{31.6} = 17.21 \therefore k = 24.4\,;\ m = 7.2.$

$\log \cos B = \log 7.2 - \log 12.76 + 10$

$= .85733 - 1.10585 + 10 = 9.75148 \therefore B = 55° 38' 57''.$

$\log \cos A = \log 24.4 - \log 26.58 + 10$

$= 1.38739 - 1.42455 + 10 = 9.96284 \therefore A = 23° 22'$

$C = 180° - (A + B) = 100° 59' 3''.$ As a check we may use (**22**); thus :—

$\log b$	$= 1.42455$	$\log c$	$= 1.49969$
$\log \sin B$	$= 9.91677$	$\log \sin C$	$= 9.99197$
log modulus	$= 1.50778$	or,	1.50772

results agreeing to the fourth decimal place.

4. Given $b = .0079$, $c = .0124$, $A = 37° 22' 17''$.

In ACD (Fig. 7),

$\log \sin A$	$= 9.78318$	$\log CD$	$= \bar{3}.68081$
$\log .0079$	$= \bar{3}.89763$	$\log DB$	$= \bar{3}.78675$
$\log CD$	$= \bar{3}.68081$	$\log \tan B$	$= 9.89406$
		$\therefore \quad B$	$= 38° 4' 48''$
$\log \cos A$	$= 9.90021$	$\log CD$	$= \bar{3}.68081$
$\log .0079$	$= \bar{3}.89763$	$\log \sin B$	$= 9.79012$
$\log AD$	$= \bar{3}.79784$	$\log a$	$= \bar{3}.89069$
AD	$= .00628$	a	$= .00777$
$\therefore DB$	$= .00612$		

We may again check by use of (**22**), thus:—

$\log a$	$= \bar{3}.89069$	$\log b$	$= \bar{3}.89763$
$\log \sin A$	$= 9.78318$	$\log \sin B$	$= 9.79012$
log modulus	$= \bar{2}.10751$	log modulus	$= \bar{2}.10751$

results agreeing exactly.

CHAPTER III.

FUNCTIONS OF THE SUM AND OF THE DIFFERENCE OF TWO ANGLES. MISCELLANEOUS FORMULÆ.

23. Sine and Cosine of the Sum.—To show that

(1) $\sin(x+y) = \sin x \cos y + \cos x \sin y$

(2) $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

Let OA, OB, and OC be any three lines whatever, forming the angles AOB and BOC (Figs. 11, 12). Let $AOB = x$ and $BOC = y$. From any point of OC, as F, drop $FE \perp OA$ and $FP \perp OB$, and from P drop $PD \perp OA$ and $PG \perp EF$.

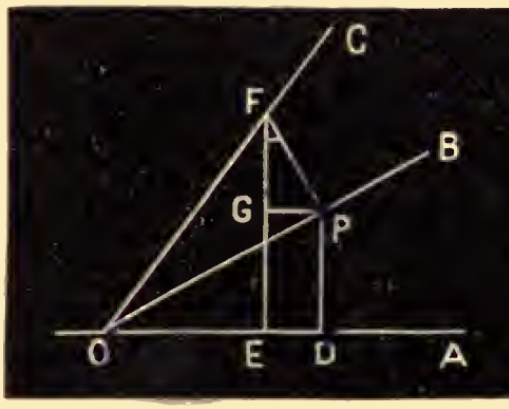


Fig. 11.

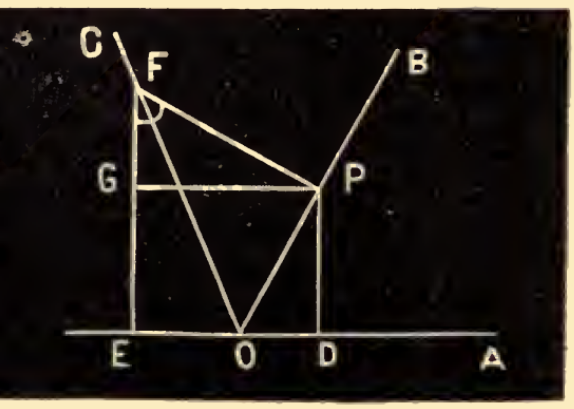


Fig. 12.

Then $PD : OP = \sin x = PG : FP$ (similar triangles), and $OD : OP = \cos x = FG : FP$; $\sin y = FP : OF$, $\cos y = OP : OF$, $\sin(x+y) = FE : OF$, and $\cos(x+y) = OE : OF$.

From the figures, $EF = EG + GF = PD + GF$, *i. e.*,

$$\frac{EF}{OF} = \frac{PD}{PO} \cdot \frac{PO}{OF} + \frac{FG}{FP} \cdot \frac{FP}{OF}; \text{ i. e.,}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \text{(I).}$$

Again, (Fig. 11), $OE = OD - GP$, whence

$$\frac{OE}{OF} = \frac{OD}{OP} \cdot \frac{OP}{OF} - \frac{GP}{FP} \cdot \frac{FP}{OF}; \text{ i. e.,}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \text{(II).}$$

The same formulæ may be deduced from figure 12, if it be noted that in this case OE is negative.

24. Sine and Cosine of the Difference.—To show that

(1) $\sin(x-y) = \sin x \cos y - \cos x \sin y$

(2) $\cos(x-y) = \cos x \cos y + \sin a \sin y.$

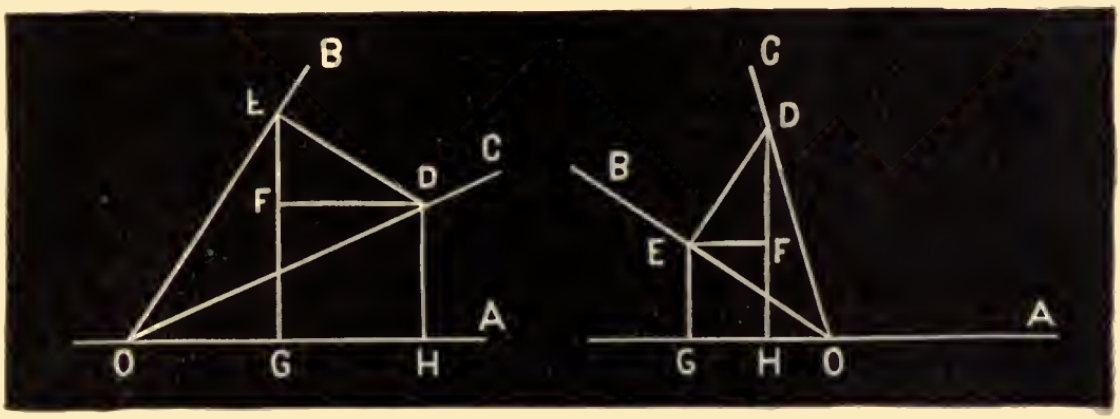


Fig. 13. Fig. 14.

Let AOB (Fig. 13) be x, and BOC (laid off backward, *i. e.*, in a *negative* direction), be $-y$, so that AOC is $x-y$. From any point, D, of OC, draw DH $\perp$ OA, and DE $\perp$ OB. Draw EG $\perp$ OA, and DF $\perp$ EG. Then EG : OE = FD : ED $= \sin x$, OG : OE = EF : ED $= \cos x$, DE : OD $= \sin y$, OE : OD $= \cos y$, DH : OD $= \sin(x-y)$, and OH : OD $= \cos(x-y)$. From the figure we see that DH = EG − EF, whence

$$\frac{DH}{OD} = \frac{EG}{OE}\cdot\frac{OE}{OD} - \frac{EF}{ED}\cdot\frac{ED}{OD}, \text{ i. e.,}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y \qquad \text{(III).}$$

Again, OH = OG + FD, whence

$$\frac{OH}{OD} = \frac{OG}{OE}\cdot\frac{OE}{OD} + \frac{FD}{ED}\cdot\frac{ED}{OD}, \text{ i. e.,}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y \qquad \text{(IV).}$$

In Fig. 14, both x and $x-y$ are obtuse.

25. Tangents.—Since $\tan(x+y) = \dfrac{\sin(x+y)}{\cos(x+y)}$, we obtain, by substitution from (**31**), $\tan(x+y) = \dfrac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y,}$

The student should notice that no supposition has been made concerning the direction of OB and OC, and that, therefore, the formulæ are true for all angles.

or, dividing both numerator and denominator by $\cos x \cos y$, (remembering that $\frac{\sin x}{\cos x} = \tan x$, etc.),

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \text{(V)}.$$

By a similar process we obtain $\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$ (VI).

34. COTANGENTS.—Since $\cot(x+y) = \frac{\cos(x+y)}{\sin(x+y)}$, we obtain, by substitution (**31**), $\cot(x+y) = \frac{\cos x \cos y - \sin x \sin y}{\sin x \cos y + \cos x \sin y}$; or, dividing both terms of the fraction by $\sin x \sin y$,

$$\cot(x+y) = \frac{\cot x \cot y - 1}{\cot y + \cot x} \qquad \text{(VII)}.$$

Similarly we obtain $\cot(x-y) = \frac{\cot x \cot y - 1}{\cot y - \cot x}$ (VIII).

35. EXERCISE.—Find (**31** – **34**) the functions of $(90 \pm x)$, $(180° \pm x)$, $(270° \pm x)$, $(360° \pm x)$. The results are given in Table II. Observe that if the number of quadrants involved be *odd* the function changes *name*, and not otherwise. Thus, $\tan(270° + x) = -\cot x$; but $\tan(180° + x) = \tan x$. The sign is determined by the quadrant in which the angle would lie, if x were less than 90°.

Table II.

FUNCTION.	$90° + x$	$90° - x$	$180° + x$	$180° - x$	$270° + x$	$270° - x$	$360° + x$	$360° - x$
SINE.	$\cos x$	$\cos x$	$-\sin x$	$\sin x$	$-\cos x$	$-\cos x$	$\sin x$	$-\sin x$
COSINE.	$-\sin x$	$\sin x$	$-\cos x$	$-\cos x$	$\sin x$	$-\sin x$	$\cos x$	$\cos x$
TANGENT.	$-\cot x$	$\cot x$	$\tan x$	$-\tan x$	$-\cot x$	$\cot x$	$\tan x$	$-\tan x$
COTANGENT.	$-\tan x$	$\tan x$	$\cot x$	$-\cot x$	$-\tan x$	$\tan x$	$\cot x$	$-\cot x$
SECANT.	$-\csc x$	$\csc x$	$-\sec x$	$-\sec x$	$\csc x$	$-\csc x$	$\sec x$	$\sec x$
COSECANT.	$\sec x$	$\sec x$	$-\csc x$	$\csc x$	$-\sec x$	$-\sec x$	$\csc x$	$-\csc x$

36. Functions of Double Angles and Half Angles. If in (**31** – **4**) we make $y = x$, we obtain,

$$\sin(x+y) = \sin(x+x) = \sin 2x = 2\sin x \cos x \qquad (A)$$

$$\cos(x+y) = \cos(x+x) = \cos 2x = \cos^2 x - \sin^2 x$$
$$= 2\cos^2 x - 1 = 1 - 2\sin^2 x \qquad (B)$$

$$\tan(x+x) = \tan 2x = \frac{2\tan x}{1-\tan^2 x} \qquad (C)$$

$$\cot(x+x) = \cot 2x = \frac{\cot^2 x - 1}{2\cot x} \qquad (D)$$

Again, from (B), we have $1 - \cos 2x = 2\sin^2 x$. Let $x = \frac{a}{2}$, whence $2x = a$. Then $2\sin^2 \frac{a}{2} = 1 - \cos a$, whence $\sin \frac{a}{2} = \sqrt{\frac{1}{2}(1-\cos a)}$ (A′). From (B) we have, also, $2\cos^2 x = 1 + \cos 2x$, or, if $x = \frac{a}{2}$, $2\cos^2 \frac{a}{2} = 1 + \cos a$, whence $\cos \frac{a}{2} = \sqrt{\frac{1}{2}(1+\cos a)}$ (B′). From (A′) and (B′) we derive, at once,

$$\tan \frac{a}{2} = \sqrt{\frac{1-\cos a}{1+\cos a}} \quad (C'); \text{ and } \cot \frac{a}{2} = \sqrt{\frac{1+\cos a}{1-\cos a}} \quad (D').$$

These are formulæ of great importance.

28. Adding (I) and (III), member to member, we obtain $\sin(x+y) + \sin(x-y) = 2\sin x \cos y$ (1). Let $x+y=m$ and $x-y=k$, whence $x = \frac{1}{2}(m+k)$; $y = \frac{1}{2}(m-k)$. Then (1) becomes $\sin m + \sin k = 2\sin \frac{1}{2}(m+k)\cos \frac{1}{2}(m-k)$ (2);

Similarly,

$$\sin m - \sin k = 2\cos \tfrac{1}{2}(m+k)\sin \tfrac{1}{2}(m-k) \qquad (3);$$

$$\cos m + \cos k = 2\cos \tfrac{1}{2}(m+k)\cos \tfrac{1}{2}(m-k) \qquad (4);$$

$$\cos m - \cos k = -2\sin \tfrac{1}{2}(m+k)\sin \tfrac{1}{2}(m-k) \qquad (5).$$

29. From (2) – (4), (28) we deduce, by direct division,

$$\frac{\sin m + \sin k}{\sin m - \sin k} = \frac{2\sin \frac{1}{2}(m+k)\cos \frac{1}{2}(m-k)}{2\cos \frac{1}{2}(m+k)\sin \frac{1}{2}(m-k)} = \frac{\tan \frac{1}{2}(m+k)}{\tan \frac{1}{2}(m-k)} \qquad (1);$$

and similarly,

$$\frac{\cos m - \cos k}{\cos m + \cos k} = -\tan \tfrac{1}{2}(m+k)\tan \tfrac{1}{2}(m-k) \qquad (2);$$

$$\frac{\sin m + \sin k}{\cos m + \cos k} = \tan \tfrac{1}{2}(m+k) \quad (3); \quad \frac{\sin m - \sin k}{\cos m - \cos k} = \cot \tfrac{1}{2}(m-k) \quad (4).$$

30. The formulæ of (28, 29) enable us to adapt formulæ involving *sums* or *differences* of functions to logarithmic calculation.

CHAPTER IV.

ADDITIONAL FORMULÆ FOR PLANE TRIANGLES.

40. *Any side of a plane triangle equals the sum of the products of the remaining sides into the cosines of the respective angles included between them and the first side.*

Thus (in Fig. 7), if CD $\perp$ AB, AD = $b \cos A$, and BD = $a \cos B$, whence $c = AD + DB = b \cos A + a \cos B$. If (as in Fig. 8), CD fall without the triangle, $c = AD - DB$. But DB = $a \cos CBD = a \cos (180° - B) = -a \cos B$

and we have as before, $c = b \cos A + a \cos B$ (1)

Similarly $b = c \cos A + a \cos C$ (2)

$a = b \cos C + c \cos B$ (3)

41. *The square of any side of a triangle equals the sum of the squares of the other two sides, minus twice the product of these sides into the cosine of their included angle.*

From (**40**), (2), by transposing and squaring, we obtain

$a^2 \cos^2 C = b^2 - 2bc \cos A + c^2 \cos^2 A$

From (**22**) $a^2 \sin^2 C = c^2 \sin^2 A$

Adding, $a^2 = b^2 - 2bc \cos A + c^2$ (1)

Similarly, $b^2 = a^2 + c^2 - 2ac \cos B$ (2)

$c^2 = b^2 + a^2 - 2ab \cos C$ (3)

42. From (1) – (3), (**41**) we deduce

$$\left.\begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \quad (1) \\ \cos B &= \frac{c^2 + a^2 - b^2}{2ac} \quad (2) \\ \cos C &= \frac{a^2 + b^2 - c^2}{2bc} \quad (3) \end{aligned}\right\} \textbf{(A)}$$

These formulæ, not being adapted for logarithmic computation, are usually transformed as follows :—

Subtracting each member of (1) from unity, we have

$$1-\cos A=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2cb-b^2-c^2+a^2}{2cb}=\frac{a^2-(b-c)^2}{2bc};\quad \text{or,}$$

since $1-\cos A=2\sin^2\frac{A}{2}$, $2\sin^2\frac{A}{2}=\frac{(a-b+c)(a+b-c)}{2bc}$.

Let $2s=a+b+c$; then $s-b=\frac{1}{2}(a+c-b)$, and $s-c=\frac{1}{2}(a+b-c)$, so that

$$\left.\begin{aligned}\sin\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{bc}}\quad(1);\\ \text{similarly,}\quad \sin\frac{B}{2}&=\sqrt{\frac{(s-a)(s-c)}{ac}}\quad(2);\\ \sin\frac{C}{2}&=\sqrt{\frac{(s-a)(s-b)}{ab}}\quad(3).\end{aligned}\right\}(\mathbf{B})$$

Adding both members of the equations (**A**) to unity we obtain, by similar reductions, formulæ (**C**).

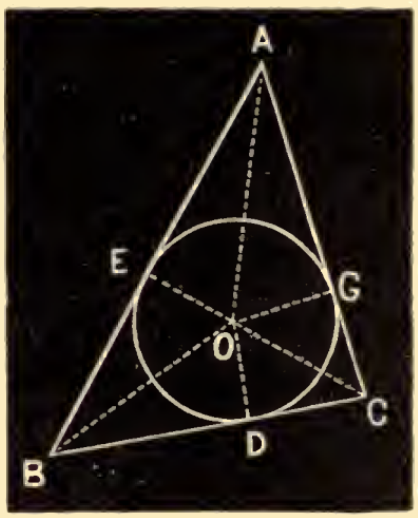
$$\left.\begin{aligned}\cos\tfrac{1}{2}A&=\sqrt{\frac{s(s-a)}{bc}}\quad(1);\\ \cos\tfrac{1}{2}B&=\sqrt{\frac{s(s-b)}{ac}}\quad(2);\\ \cos\tfrac{1}{2}C&=\sqrt{\frac{s(s-c)}{ab}}\quad(3).\end{aligned}\right\}(\mathbf{C})$$

From (**B**) and (**C**) we obtain, by direct division, (**D**).

$$\left.\begin{aligned}\tan\tfrac{1}{2}A&=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\quad(1);\\ \tan\tfrac{1}{2}B&=\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}\quad(2);\\ \tan\tfrac{1}{2}C&=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}\quad(3).\end{aligned}\right\}(\mathbf{D})$$

Of these three sets, (**B**), (**C**), (**D**), the FIRST should be used *only when the angles are small;* the SECOND should *never be used*

with small angles. The formulæ (**D**) are the *best for general use*, being accurate for all angles. These formulæ furnish the means of solving Case IV (**21**).



34. Radius of Inscribed Circle (r). Area.—Let ABC be any triangle. The bisectrices of the angles meet in the centre of the inscribed circle. Also $AE = s - CB$, $BD = s - AC$, etc. Hence, in the right triangle AOE,

$$\tan OAE = \tan \tfrac{1}{2} A = \frac{OE}{s-a} = \frac{r}{s-a} = (\mathbf{33, D}),$$

$$\sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{1}{s-a}\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\therefore r = \tan \tfrac{1}{2} A.\,(s-a) = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Again, as the area of a triangle equals one-half the perimeter multiplied by the radius of the inscribed circle, the area of ABC

$$= s\sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{s\,(s-a)(s-b)(s-c)}.$$

35. Another expression for the area is "*one-half the product of any two sides into the sine of the included angle.*"

Thus (Figs. 7, 8), area $ABC = \frac{1}{2} AB \,.\, CD = \frac{1}{2} AB \,.\, b \sin A = \frac{1}{2} bc \sin A$. This is equivalent to $bc \sin \frac{1}{2} A \cos \frac{1}{2} A$.

36. *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Let ABC be the triangle. By (**22**), $a : b :: \sin A : \sin B$. By composition and division, $a + b : a - b :: \sin A + \sin B : \sin A - \sin B ::$ (**29**), $\tan \frac{1}{2}(A + B) : \tan \frac{1}{2}(A - B)$. Q. E. D.

Case II (**21**) may be solved by means of this proposition.

CHAPTER V.

SPHERICAL TRIGONOMETRY.

36. Spherical Trigonometry treats of the solution of spherical triangles, or the relations between the face-angles and diedrals of the triedrals, whose common vertex is the centre of the sphere, and which intercept these triangles on the surface. As the sides of these triangles are expressed in degrees, or in radial measure, the solution of the triangle gives, not the absolute lengths of the sides, but only the ratios existing between them.

To determine the absolute lengths we must know the radius. This premised, any three parts of a spherical triangle suffice to determine the other three. Two parts are of the **same species** when they lie in the same quadrant, and of **different species** when they lie in different quadrants. We consider only triangles in which no part exceeds 180°; this convention, though not absolutely necessary, simplifies our calculations without causing any material loss of generality.

37. Right Triangles. Napier's Parts. Napier's Rules.—To express, in two general rules, the various formulæ required in solving a right triangle, Napier adopted the following convention: In any right triangle the two sides about the right angle, the complements of the two angles opposite them, and the complement of the hypothenuse, are called the five **circular parts.** If any three of these be considered, one will be *adjacent* to both the others, or one will be *separated* from each of the others by the intervention *of a circular part.* A part *adjacent* to two others is called a *middle* part, and the other two are called **adjacent extremes.** A part *separated* from two others is called a *middle,* and the other two are called **opposite extremes.**

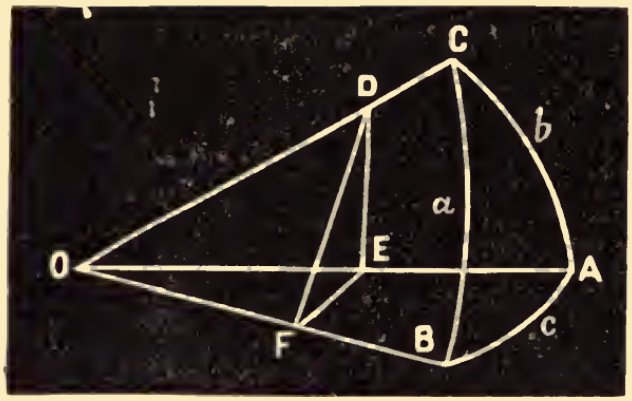


Fig. 15.

Napier's Rules.—I. *The* SINE *of the middle part equals the product of the* COSINES *of the* OPPOSITE *extremes.*

Taking the circular parts successively as middles, we have the following equations to establish:—

(1) $\sin b = \sin a \sin B$

(2) $\sin c = \sin a \sin C$

(3) $\cos a = \cos b \cos c$

(4) $\cos B = \cos b \sin C$

(5) $\cos C = \cos c \sin B$

Let ABC be the right triangle, A being the right angle. Connect the vertices with the centre (O) of the sphere. From any point D, of OC, drop DE $\perp$ OA, and from E draw EF $\perp$ OB. Then DF must be perpendicular to OB, and DFE, the rectilineal angle of the diedral, OB, will equal the spherical angle B. DEO, EOF, DOF, and DFE, are respectively triangles of reference for b, c, a, and B.

(1) As $\sin b = DE:OD$, $\sin a = DF:OD$, and $\sin B = DE:DF$, we have, at once,

$$\frac{DE}{OD} = \frac{DF}{OD} \cdot \frac{DE}{DF}, \text{ or}$$

$$\sin b = \sin a \sin B. \quad \text{Q. E. D.}$$

The lettering of the figure being purely arbitrary, so far as B and C (and therefore b and c), are concerned, (2) follows at once from (1).

(3) As $$\frac{OF}{OD} = \frac{OE}{OD} \cdot \frac{OF}{OE},$$

we have, at once, $\cos a = \cos b \cos c$. Q. E. D.

(4) To show that $\cos B = \cos b \sin C$. We cannot get sin C directly from the figure, so we replace it by $\frac{\sin c}{\sin a}$ (2), as sin c and

$\sin a$ can be obtained directly. We must prove, then, that $\cos B = \cos b \frac{\sin c}{\sin a}$; i. e., $\cos B = \cos b \sin c \csc a$.

But we have, directly, $\frac{EF}{FD} = \frac{OE}{OD} \cdot \frac{EF}{OE} \cdot \frac{OD}{FD}$; i. e.,

$\cos B = \cos b \sin c \csc a = \cos b \sin C$. Q. E. D.

II. *The* SINE *of the middle part equals the product of the* TANGENTS *of the* ADJACENT *extremes.*

We are to prove that:—

(1) $\sin b = \tan c \cot C$

(2) $\sin c = \tan b \cot B$

(3) $\cos B = \tan c \cot a$

(4) $\cos C = \tan b \cot a$

(5) $\cos a = \cot B \cot C$

(1) From (I, (1)) we have, $\sin b = \sin a \sin B$; from (I, (2)) $\sin a = \frac{\sin c}{\sin C}$, and from (I,(5)) $\sin B = \frac{\cos C}{\cos c}$; whence

$$\sin b = \frac{\sin c}{\sin C} \cdot \frac{\cos C}{\cos c} = \frac{\sin c}{\cos c} \cdot \frac{\cos C}{\sin C} = \tan c \cot C. \text{ Q. E. D.}$$

(2) Similarly $\sin c = \tan b \cot B$.

(3) From (I,(4)) $\cos B = \cos b \sin C$, and as

$$\cos b = \frac{\cos a}{\cos c}, \text{ and } \sin C = \frac{\sin c}{\sin a}, \text{ we have}$$

$$\cos B = \frac{\cos a}{\cos c} \cdot \frac{\sin c}{\sin a} = \tan c \cot a. \quad \text{Q. E. D.}$$

(4) Similarly, $\cos C = \tan b \cot a$.

(5) Finally, as (I) $\cos a = \cos b \cos c$, and $\cos b = \frac{\cos B}{\sin C}$,

and $\cos c = \frac{\cos C}{\sin B}$, $\cos a = \frac{\cos B}{\sin C} \cdot \frac{\cos C}{\sin B} = \cot B \cot C$. Q. E. D.

38. Napier's Rules suffice for the solution of all problems involving only right triangles. Slight difficulties may, however, present themselves in actual practice, when we wish to decide whether an angle (arc) whose functions are numerically determined, lies in the first, or in the second, quadrant.

38. Determination of Species.—If, in any investigation, a part be determined by means of the cosine, tangent, or cotangent, the *sign* of the function will indicate the quadrant in which the part is to be found. If, however, the part be determined by means of the *sine*, an ambiguity presents itself, as the sine is positive in both the first and second quadrants. In this event we note that:—

I. *An oblique angle and the side opposite are always of the same species.*

From (37) $\sin C = \frac{\cos B}{\cos b}$; as $\sin C$ is necessarily positive, $\cos B$ and $\cos b$ must agree in sign; i. e., B and b must lie in the same quadrant.

II. *If the hypothenuse exceed* 90°, *the remaining sides, and consequently the oblique angles, are of* DIFFERENT *species.*

From (37), $\cos a = \cos b \cos c$. Hence, if $a > 90°$, $\cos a$ is negative, and therefore $\cos b$ and $\cos c$ are of different signs; i. e., b and c, and therefore, by (I), B and C, lie in different quadrants. If, however, $a < 90°$, $\cos a$ is positive, and hence $\cos b$ and $\cos c$ agree in sign, and therefore b and c (and likewise B and C), lie in the same quadrant. Hence

III. *If the hypothenuse be less than* 90°, *the remaining sides, and therefore the oblique angles, are of the* SAME *species.*

IV. *If a side (other than the hypothenuse), and the angle opposite be given, there will be* TWO *solutions, if the sine of the side be less than the sine of the angle;* ONE *solution if the sine of the side equal the sine of the angle, and* NO *solution if the sine of the side exceed the sine of the angle.*

From (37), $\sin a = \frac{\sin b}{\sin B}$; hence (4) $\sin b$ cannot exceed $\sin B$. If $\sin b = \sin B$, $\sin a = 1$, $a = 90°$, and the triangle is birectangular. If $\sin b < \sin B$, $\sin a < 1$, but positive, and a may lie either in the first or in the second quadrant.

40. Quadrantal Triangles, i. e., triangles in which at least one side is a quadrant, may be solved by Napier's Rules,

as the polar of every quadrantal triangle must be a right triangle.

41. Example.

In a right triangle, $b = 38° 16'$; $c = 120° 35'$; solve.

From (37) $\cos a = \cos b \cos c$, whence

$$\log \cos a = \log \cos b + \log \cos c - 10$$
$$= 9.89495 + 9.73121 - 10 = 9.62616.$$

$b < 90°$ and $c > 90°$; $\therefore \cos b$ is $+$, and $\cos c$ is $-$, so that $\cos a$ is negative, and $a > 90°$; $\therefore a = 155° 0' 47''$. To avoid using a we compute C by (37, II, 1), which gives

$$\log \cot C = \log \sin b - \log \tan c$$
$$= 9.79192 - 10.19442 + 10 = .5975.$$

$c > 90°$; $C > 90°$, (39) $\therefore C = 111° 35' 40''$.

Finally, $\log \cot B = \log \sin c - \log \tan b$

$$= 9.92563 - 9.89697 = .02866$$

whence $B = 43° 06' 40''$.

42. The formulæ of Spherical Trigonometry correspond closely to those of Plane Trigonometry; differences arise from the fact that the sides of plane triangles are dealt with directly, while the sides of spherical triangles are dealt with only by means of their functions. Analogies deserve attention so far as they aid in fixing formulæ in mind. Table III suggests analogies between the formulæ for solving Plane Right triangles and those employed in solving Spherical Right triangles. More striking analogies will present themselves in the case of oblique triangles.

Table III.

Right Plane Triangles.	*Right Spherical Triangles.*
$\sin B = \frac{b}{a}$; $\sin C = \frac{c}{a}$.	$\sin B = \frac{\sin b}{\sin a}$; $\sin C = \frac{\sin c}{\sin a}$.
$\cos B = \frac{c}{a}$; $\cos C = \frac{b}{a}$.	$\cos B = \frac{\tan c}{\tan a}$; $\cos C = \frac{\tan b}{\tan a}$.
$\tan B = \frac{b}{c}$; $\tan C = \frac{c}{b}$.	$\tan B = \frac{\tan b}{\sin c}$; $\tan C = \frac{\tan c}{\sin b}$.
$\cot B = \frac{c}{b}$; $\cot C = \frac{b}{c}$.	$\cot B = \frac{\sin c}{\tan b}$; $\cot C = \frac{\sin b}{\tan c}$.

43. Oblique Triangles.—I. *Given two sides and the included angle*, e. g., *b*, *c*, *A*. From the extremity of one known side drop BD, perpendicular to the other. Solve (**37**) ABD. If AD $<$ *b*, the perpendicular falls within the triangle; if AD $>$ *b*, the perpendicular falls without. In either case, *p* and *m* being known, BCD may be solved. Finally, the angle ABC = ABD + DBC, or ABD − DBC, according as DB falls within or without the triangle.

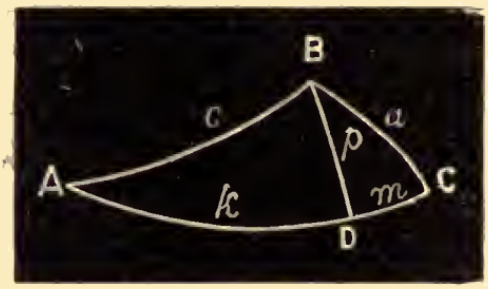


Fig. 16.

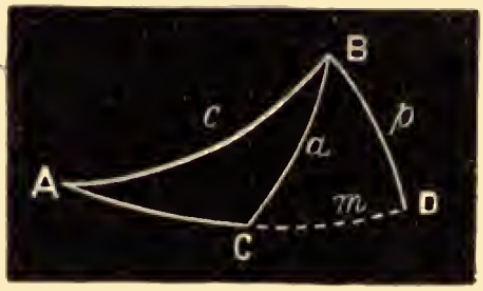


Fig. 17.

II. *Given two angles and the included side*, e. g., *B*, *A*, *c*. Drop the perpendicular, BD, from one extremity of the known side, to *b*. Solve the triangle ABD as in **I**. If ABD $>$ B the perpendicular falls without; if ABD $<$ B, the perpendicular falls within the triangle. *p* and DBC being known, the triangle DBC may be solved. Finally, AC = *k* + *m*, or *k* − *m*, according as BD falls within or without ABC.

III. *Given two sides, and an angle opposite one of them;* e. g., *a*, *c*, *A*. Drop BD $\perp$ *b*, the unknown side. Solve ABD (**37**); in DBC the sides *a* and BD will then be known, and DBC may be solved. But as two equal arcs may be drawn from a point to an arc of a great circle, *a* may occupy either of two positions, one on the right, and the other on the left, of BD. Hence (compare (**21**) III), there may be two solutions. There

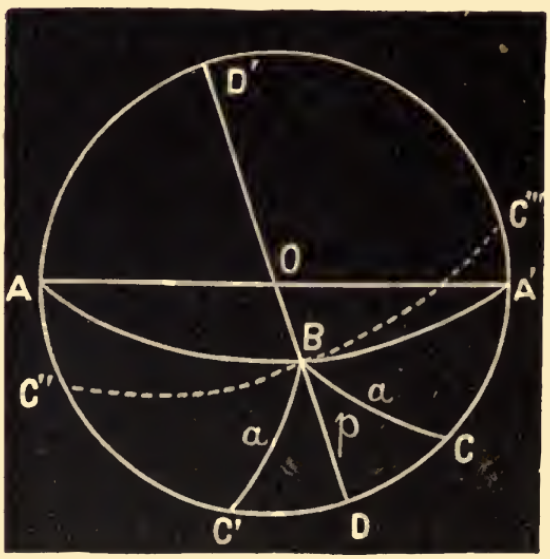


Fig. 18.

will evidently be two solutions if a be intermediate in value between p and c, and *also* between p and $180° - c$.

I. If $A < 90°$, $a > p$, and $< c$, but $a > 180° - c$, there can be but one solution, as ABC″; for, though a might occupy a second position BC‴, as far to the right as BC″ is to the left, of BD, C‴ would fall beyond A′, since BC‴ $>$ BA′, and the arc AC‴ would exceed 180°, so that we could not (**36**) consider the triangle ABC‴.

II. If $A < 90°$, $a > p$, and $a > c$, but $a < 180° - c$, a must fall to the right of the perpendicular, and there will be but one solution.

III. If $A > 90°$, the conditions I, II, must of course be changed by reversing the signs of inequality.

IV. If $a = p$, ABC and ABC′ coincide with ABD.

V. If $a < p$, $A < 90°$, there will be no solution, as, when $A < 90°$, the perpendicular is the shortest arc (of a great circle) from B to ADA′.

VI. If $a > p$, $A > 90°$, there will be no solution, for if $A > 90°$ the perpendicular is the longest arc of a great circle that can be drawn from B to ADA′.

The number of solutions should be determined as soon as BD is computed.

IV. *Given two angles and a side opposite one of them,* e. g., A, C, a. The polar or supplemental triangle may be solved as in **III.** The angles and sides of ABC may then be obtained by taking the supplements respectively of the sides and angles of the polar.

V. *Given the sides: a, b, c.* From any vertex, as B, draw

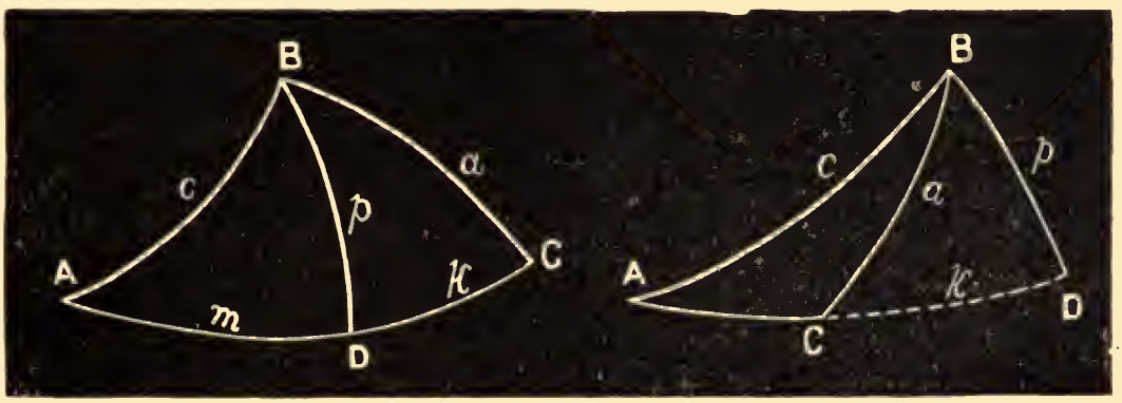


Fig 19.

Fig. 20.

BD, $\perp b$. Let AD $= m$, and CD $= k$. Then, (**37**),
$\cos c = \cos m \cos p$ (1), and $\cos a = \cos k \cos p$ (2), whence
$\cos c : \cos a = \cos m : \cos k$, and, by division and composition,
$\cos c - \cos a : \cos c + \cos a = \cos m - \cos k : \cos m + \cos k$, or
(**29**,2), $\tan \frac{1}{2}(c+a).\tan \frac{1}{2}(c-a) = \tan \frac{1}{2}(m+k).\tan \frac{1}{2}(m-k)$(3).
From (3) either $m+k$ or $m-k$ may be determined if the other be known. If BD fall within the triangle, $m+k=b$; if BD fall without, $m-k=b$ (numerically). The symmetry of (3) shows that either $\frac{1}{2}(m+k)$ or $\frac{1}{2}(m-k)$ may be assumed equal to $\frac{1}{2}b$ in the first instance, the other being

$$\tan^{-1} \frac{\tan \frac{1}{2}(c+a) \tan \frac{1}{2}(c-a)}{\tan \frac{1}{2}b} \qquad (4).$$

Adding this latter value, (4), to $\frac{1}{2}b$, we obtain the greater segment; subtracting (4) from $\frac{1}{2}b$ we obtain the other segment. If either segment exceed b, the perpendicular must fall without, and b must be the numerical *difference* of the segments.

The signs of the functions in (3) are important;
$\tan \frac{1}{2}(c+a)$ will be + if $a+c < 180°$, and − if $c+a > 180°$;
$\tan \frac{1}{2}(c-a)$ is + if $c > a$, and − if $c < a$; for $c-a$ cannot exexceed 180°. $\tan \frac{1}{2}b$ is +.

If the tangent (4) be negative it is the tangent of half the numerical difference of the segments, and $m < k$. For $\tan \frac{1}{2}(m+k)$ is always +. For, the feet of the two perpendiculars that can be dropped from B to AC... are 180° apart, and if they both fall without the triangle we take D as the foot of the nearer. Hence $k+m$ is always less than 180°.

k and m being determined, ABD and CBD may be solved, (**37**), and thus all parts of ABC obtained.

From (3) we derive the following important theorem (compare **21**, IV):—

If from any vertex of a spherical triangle, a perpendicular be dropped to the opposite side, the tangent of half the sum of the segments of this side is to the tangent of half the sum of the other two sides as the tangent of half the difference of the sides is to the tangent of half the difference of the segments.

VI. *Given the angles* A, B, C. Solve the polar by **V.** The supplements of its angles are the sides of the original triangle.

The solutions **III, IV,** may be facilitated by use of the theorem, "*The sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*"

Thus (Figs. 19, 20), $\sin p = \sin a \sin C$; also, in ABD, $\sin p = \sin c \sin A$, whence $\sin a \sin C = \sin c \sin A$ or,

$\sin a : \sin c :: \sin A : \sin C$.

This theorem may also be used as a "check" on the other solutions of this article.

SECTION II.—ADDITIONAL FORMULÆ.

44. Right Triangles.—When B or C is small, computation by means of the cosine should (**18**) be avoided. We may use the following formula:—

$$\tan^2\frac{B}{2} = \frac{1-\cos B}{1+\cos B} = \frac{1-\tan c \cot a}{1+\tan c \cot a} = \frac{\tan a - \tan c}{\tan a + \tan c} = \frac{\sin (a-c)}{\sin (a+c)},$$

$$\text{whence } \tan \tfrac{1}{2} B = \sqrt{\frac{\sin (a-c)}{\sin (a+c)}} \qquad \text{(I)}.$$

Again, if a be near 90°, we may use the formula

$$\tan \frac{a}{2} = \sqrt{-\frac{\cos (B+C)}{\cos (B-C)}} \text{ [proof similar to proof of (I)]} \qquad \text{(II)}.$$

If b or c be near 90°, it will be better not to compute by means of the sine.

45. Oblique Triangles.—In any spherical triangle.

The cosine of any side equals the product of the cosines of the other two, plus the product of the sines of these sides into the cosine of the included angle.

To prove (Figs. 16, 17) $\cos b = \cos a \cos c + \sin a \sin c \cos B$. (1)

By (**37**) $\left.\begin{array}{l} \cos a = \cos m \cos p\,; \quad \cos c = \cos k \cos p, \\ \sin a = \sin p \div \sin C\,; \quad \sin c = \sin p \div \sin A \end{array}\right\}$ (2)

Again, (**37**) $\cos B = \cos ABD \cos DBC - \sin ABD \sin DBC$.

By (**37**) $\cos ABD = \sin A \cos k$; $\sin ABD = \sin k \div \sin c$;

$\cos DBC = \sin C \cos m$; $\sin DBC = \sin m \div \sin a$

Hence $\cos B = \cos m \cos k \sin A \sin C - \dfrac{\sin m \sin k}{\sin a \sin c}$. (3)

Substituting the values (2),(3) in the second member of (1), we obtain

$\cos m \cos k \cos^2 p + \cos m \cos k \sin^2 p - \sin k \sin m$, i. e.,

$\cos m \cos k - \sin m \sin k$, which equals $\cos b$, or,

$\cos a \cos c + \sin a \sin c \cos B = \cos b$. Q. E. D.

If p fall without, (Fig. 17), $B = ABD - DBC$, and $b = k - m$, and the reduction is similar to that just indicated. For a and c we have

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \quad (4),$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C \quad (5).$$

46. From (1), (4), (5), we deduce

$$\left.\begin{aligned} \cos A &= \frac{\cos a - \cos b \cos c}{\sin b \sin c} && (1);\\ \cos B &= \frac{\cos b - \cos a \cos c}{\sin a \sin c} && (2);\\ \cos C &= \frac{\cos c - \cos a \cos b}{\sin a \sin b} && (3). \end{aligned}\right\} \textbf{(A)}$$

These formulæ, like their analogues, (**33**) are ill adapted for logarithmic computation. We therefore transform them, as follows: From (1) we deduce

$$1 - \cos A = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\sin b \sin c + \cos b \cos c - \cos a}{\sin b \sin c},$$

or, $$\sin^2 \tfrac{1}{2} A = \frac{\cos(b-c) - \cos a}{2 \sin b \sin c} = \frac{2 \sin \tfrac{1}{2}(a+b-c) \sin \tfrac{1}{2}(a+c-b)}{2 \sin b \sin c},$$

or, placing $2s = a+b+c$, $\sin^2 \tfrac{1}{2} A = \dfrac{\sin(s-c) \sin(s-b)}{\sin b \sin c}$; $\therefore$

$$\left.\begin{aligned} \sin \tfrac{1}{2} A &= \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}} && (1);\\ \text{similarly,}\quad \sin \tfrac{1}{2} B &= \sqrt{\frac{\sin(s-a) \sin(s-c)}{\sin a \sin c}} && (2);\\ \sin \tfrac{1}{2} C &= \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}} && (3). \end{aligned}\right\} \textbf{(B)}$$

47. Adding both members of (1), (2), (3), (**A**), to unity and reducing in a manner similar to that just shown, we obtain

$$\left.\begin{aligned} \cos\tfrac{1}{2}A &= \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} && (1);\\ \cos\tfrac{1}{2}B &= \sqrt{\frac{\sin s \sin(s-b)}{\sin b \sin c}} && (2);\\ \cos\tfrac{1}{2}C &= \sqrt{\frac{\sin s \sin(s-c)}{\sin b \sin c}} && (3). \end{aligned}\right\} (\mathbf{C})$$

The tabulated cosines are inaccurate for small angles, and the sines are unsatisfactory for angles near 90°. We therefore obtain, by direct division, from (**B**) and (**C**), the tangents, (**D**), which may be satisfactorily employed for all angles.

$$\left.\begin{aligned} \tan\tfrac{1}{2}A &= \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s \sin(s-a)}} && (1);\\ \tan\tfrac{1}{2}B &= \sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s \sin(s-b)}} && (2);\\ \tan\tfrac{1}{2}C &= \sqrt{\frac{\sin(s-b)\sin(s-a)}{\sin s \sin(s-c)}} && (3). \end{aligned}\right\} (\mathbf{D})$$

If A′B′C′ be the polar of ABC,

$$\left.\begin{aligned} A' &= 180° - a,\ B' = 180° - b,\ C' = 180° - c.\\ a' &= 180° - A,\ b' = 180° - B,\ c' = 180° - C. \end{aligned}\right\}$$

In this polar then we have (**45**)

$\cos a' = \cos b' \cos c' - \sin b' \sin c' \cos A'$, etc., or, by substitution,

$$\left.\begin{aligned} -\cos A &= \cos B \cos C - \sin B \sin C \cos a && (1);\\ -\cos B &= \cos A \cos C - \sin A \sin C \cos b && (2);\\ -\cos C &= \cos A \cos B - \sin A \sin B \cos c && (3). \end{aligned}\right\}$$

From (1), (2), (3), by methods similar to those employed in (**45–6**) we obtain the formulæ (**A′**), and, adapting these to logarithmic computation, the sets (**B′**) and (**C′**), from which by direct division we obtain (**D′**).

In all these formulæ $s = \frac{1}{2}(A + B + C)$. (**B′**), (**C′**), (**D′**), may also be obtained from (**B**), (**C**), (**D**), by passing directly to the

polar triangle, as in the deduction of (**A′**) from (**A**). The general remarks on (**A**), (**B**), (**C**), (**D**), are applicable also to (**A′**), (**B′**), (**C′**), (**D′**).

$$\left.\begin{aligned} \cos a &= \frac{\cos B \cos C - \cos A}{\sin B \sin C}; \\ \cos b &= \frac{\cos A \cos C - \cos B}{\sin A \sin C}; \\ \cos c &= \frac{\cos A \cos B - \cos C}{\sin A \sin B}. \end{aligned}\right\} (\mathbf{A'})$$

$$\left.\begin{aligned} \sin \tfrac{1}{2} a &= \sqrt{\frac{-\cos S \cos (S - A)}{\sin B \sin C}}; \\ \sin \tfrac{1}{2} b &= \sqrt{\frac{-\cos S \cos (S - B)}{\sin A \sin C}}; \\ \sin \tfrac{1}{2} c &= \sqrt{\frac{-\cos S \cos (S - C)}{\sin A \sin B}}. \end{aligned}\right\} (\mathbf{B'})$$

$$\left.\begin{aligned} \cos \tfrac{1}{2} a &= \sqrt{\frac{\cos (S - B) \cos (S - C)}{\sin B \sin C}}; \\ \cos \tfrac{1}{2} b &= \sqrt{\frac{\cos (S - A) \cos (S - C)}{\sin A \sin C}}; \\ \cos \tfrac{1}{2} c &= \sqrt{\frac{\cos (S - A) \cos (S - B)}{\sin A \sin B}}. \end{aligned}\right\} (\mathbf{C'})$$

$$\left.\begin{aligned} \cot \tfrac{1}{2} a &= \sqrt{\frac{-\cos (S - B) \cos (S - C)}{\cos S \cos (S - A)}}; \\ \cot \tfrac{1}{2} b &= \sqrt{\frac{-\cos (S - A) \cos (S - C)}{\cos S \cos (S - B)}}. \\ \cot \tfrac{1}{2} c &= \sqrt{\frac{-\cos (S - A) \cos (S - B)}{\cos S \cos (S - C)}}. \end{aligned}\right\} (\mathbf{D'})$$

48. The analogies obtaining between the formulæ of Spherical Trigonometry and those of Plane Trigonometry are nowhere more interesting than at this part of our work. We duduced in (**34**) an expression for the radius of the inscribed

circle (r) in terms of the sides of a plane triangle. We have an analogous expression in the spherical triangle. Let O, (Fig. 21), be the pole of the inscribed circle of ABC. AOF, FOB, etc., are right triangles; $OAF = \frac{1}{2}A$, $OBD = \frac{1}{2}B$, etc., and $AE = AF$, $BF = BD$, $CD = CE$. Hence (**36**)

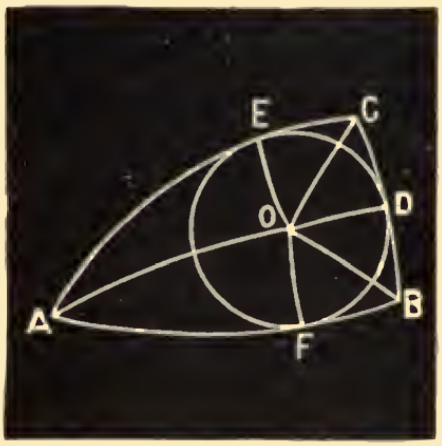


Fig. 21.

$$\tan \tfrac{1}{2} A = \frac{\tan OF}{\sin AF} = \frac{\tan r}{\sin(s-a)} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}, \text{ or}$$

$$\tan \tfrac{1}{2} A = \frac{1}{\sin(s-a)} \sqrt{\frac{\sin(s-b)\sin(s-c)\sin(s-a)}{\sin s}} = \frac{\tan r}{\sin(s-a)},$$

whence $\tan r = \sqrt{\dfrac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$, in which r is the arcual radius of the inscribed circle, and $s = \frac{1}{2}(a+b+c)$.

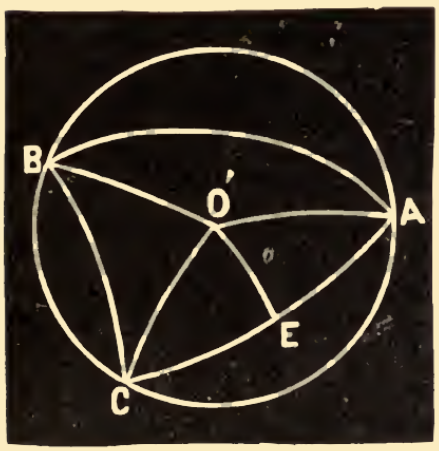


Fig. 22.

Again, let $R = O'A$ represent (Fig. 22) the arcual radius of the circumscribed circle of ABC. In any right triangle, as AO'E, we have, $\cos O'AE = \cot R \tan AE$. By consideration of the isosceles triangles AO'C, etc., we find that $O'AE = S - C$ and $AE = \frac{1}{2}C$; hence

$$\cot R = \frac{\cos(S-C)}{\tan \frac{1}{2} C} = (\mathbf{47, D'})$$

$$\cos(S-C) \sqrt{\frac{-\cos(S-A)\cos(S-B)}{\cos S \cos(S-C)}} =$$

$$\sqrt{\frac{-\cos(S-A)\cos(S-B)\cos(S-C)}{\cos S}}.$$

49. Napier's Analogies.—From (**46–7**) we readily deduce *Napier's Analogies*, viz.:—

$$\frac{\tan\frac{1}{2}(A+B)}{\cot\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\ (1);\quad \frac{\tan\frac{1}{2}(a+b)}{\tan\frac{1}{2}c}=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\ (3);$$

$$\frac{\tan\frac{1}{2}(A-B)}{\cot\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\ (2);\quad \frac{\tan\frac{1}{2}(a-b)}{\tan\frac{1}{2}c}=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\ (4).$$

To establish (1):—

$\cos\frac{1}{2}(a-b)=\cos(\frac{1}{2}a-\frac{1}{2}b)=\cos\frac{1}{2}a\cos\frac{1}{2}b+\sin\frac{1}{2}a\sin\frac{1}{2}b$; substituting for $\cos\frac{1}{2}a$, etc., their values (**46–7**), we obtain

$$\cos\tfrac{1}{2}(a-b)=\frac{\cos(S-C)-\cos S}{\sin C}\cdot\sqrt{\frac{\cos(S-B)\cos(S-A)}{\sin B\sin A}}$$

$$=\frac{\cos(S-C)-\cos S}{\sin C}\cdot\cos\tfrac{1}{2}c\ (\mathbf{47})\qquad(a).$$

Similarly, $\cos\frac{1}{2}(a+b)=\frac{\cos(S-C)+\cos S}{\sin C}\cos\frac{1}{2}c\qquad(b).$

By division,

$$\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}=\frac{\cos(S-C)-\cos S}{\cos(S-C)+\cos S}=\frac{\sin(S-\frac{1}{2}C)\sin\frac{1}{2}C}{\cos(S-\frac{1}{2}C)\cos\frac{1}{2}C}\qquad(c).$$

As $S=\frac{1}{2}(A+B+C)$, $S-\frac{1}{2}C=\frac{1}{2}(A+B)$, and we have

$$\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}=\frac{\tan\frac{1}{2}(A+B)}{\cot\frac{1}{2}C}.\quad \text{Q. E. D.}$$

The other equations may be similarly established.

50. In a similar manner we may also establish the truth of *Delambre's Analogies*, otherwise known as *Gauss's Equations*, viz.:—

$$(1)\quad \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c};$$

$$(2)\quad \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c};$$

$$(3)\quad \frac{\cos\frac{1}{2}(A+B)}{\sin\frac{1}{2}C}=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c};$$

$$(4)\quad \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}.$$

These formulæ afford solutions of cases I, II (**43**); the species of the parts should be carefully determined from the signs of the functions.

Napier's Analogies may be readily deduced, by division, from Delambre's.

CHAPTER VI.

ALGEBRAIC TRIGONOMETRY.

51. The Trigonometric functions have thus far been treated in a somewhat restricted way, from a point of view almost purely geometric. We now propose to show that the relation between the arc (or angle) and the function may be expressed algebraically, and thus that Trigonometry may be treated as a branch of pure Algebra. We will also extend our discussions to other transcendental functions of the same general form as those with which we have been dealing, and in properties and relations closely analogous to them.

52. *To develop $\sin x$ and $\cos x$ into series arranged according to the powers of x, i. e., to express algebraically the relation between x and $\sin x$ and $\cos x$ respectively.*

Whatever the relation between x and $\sin x$, let it be expressed by the identity

$$\sin x = M + N x + P x^2 + Q x^3 + R x^4 + \ldots \qquad (1)$$

in which M, N, P, Q, etc., are independent of x, and are to be determined. After we have determined them we must further show that the series is convergent.

As $\sin 0 = 0$, the second member of (1) must vanish when $x = 0$, and hence can contain no term independent of x; $\therefore M = 0$. Again, as $\sin(-x) = -\sin x$, the second member of (1) must change its sign (without changing its numerical value), when $-x$ is substituted for x. Hence it can contain no even power of x. Further, the coefficient of the first term must be 1, since for an infinitely small arc the ratio $(\sin x) : x = 1$. We must therefore modify the form of (1) by assuming a development of the form,

$$\sin x = x + Q x^3 + S x^5 + U x^7 + \ldots\ldots \qquad (2).$$

In the development of $\cos x$ no term of an odd degree can occur, for the development must remain unchanged when for x we substitute $-x$. Again, the development must contain a term independent of x and equal to unity, since, when $x=0$ we must have $\cos x=1$ (7). We assume then,

$$\cos x = 1 + R'x^2 + T'x^4 + V'x^6 + \ldots . \qquad (3).$$

R', T', V', etc., being independent of x, and to be determined.

Since (2) is an identity we may substitute $2x$ for x, thus:—

$$\sin 2x = 2\sin x \cos x = 2x + 8Qx^3 + 32Sx^5 + 128Ux^7 + \ldots . \quad (4).$$

or,
$$\sin x \cos x = x + 4Qx^3 + 16Sx^5 + 64Ux^7 + \ldots . \quad (5).$$

Multiplying (2) by (3), member by member, we obtain,

$$\sin x \cos x = x + (Q+R')x^3 + (S+T'+QR')x^5 + (U+V'+QT'+SR')x^7 + \ldots . \qquad (6).$$

Equating the second members of (5) and (6), we obtain,

$$x + (Q+R')x^3 + (S+T'+QR')x^5 + (U+V'+QT'+SR')x^7 + \ldots = x + 4Qx^3 + 16Sx^5 + 64Ux^7 + \ldots .$$

Equating coefficients, (Alg. 462), we obtain

$$\left.\begin{array}{l} Q+R'=4Q \ (a); \quad S+T'+QR'=16S \ (b); \\ U+V'+QT'+SR'=64U \ (c); \text{ etc.} \end{array}\right\} \text{(A)}$$

Again, $\sin^2 x + \cos^2 x = 1$; substituting for $\sin x$, $\cos x$, their values, (2), (3), we obtain

$$1 + (1+2R')x^2 + (R'^2+2Q+2T')x^4 + (Q^2+2S+2T'R'+2V')x^6 + (T'^2+2QS+2U+2V'R'+2W')x^8 + \ldots . = 1. \qquad (8).$$

As this is an identity we have,

$$\left.\begin{array}{l} 1+2R'=0 \ (a'); \quad R'^2+2Q+2T'=0 \ (b'); \\ Q^2+2S+2T'R'+2V'=0 \qquad (c'); \text{ etc.} \end{array}\right\} \text{(B)}$$

From (A) and (B) we obtain Q, R', S, T', etc., thus:—

$$1+2R'=0, \ \therefore \ R'=-\frac{1}{2}.$$

From (a), $R'=3Q$, $\quad \therefore \quad Q=\frac{1}{3}R'=-\frac{1}{6}=-\frac{1}{3!}$

From (b'), $\frac{1}{4}-\frac{1}{3}+2T'=0$, $\quad \therefore \quad T'=\frac{1}{24}=\frac{1}{4!}$

From (b), $S+\frac{1}{24}+\frac{1}{12}=16S$, $\quad \therefore \quad S=\frac{1}{120}=\frac{1}{5!}$

From (c'), $\frac{1}{36}+\frac{1}{60}-\frac{1}{24}=-2V'$, $\therefore\ V'=\frac{1}{720}=\frac{1}{6!}$

etc., etc.

Substituting the values of these coefficients in (2) and (3) we obtain

$$\sin x = x - \frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\ldots\ldots\ldots\ldots \quad \textbf{(A)}.$$

$$\cos x = 1 - \frac{x^2}{2}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\ldots\ldots\ldots\ldots \quad \textbf{(B)}.$$

The series **(A)** and **(B)** are convergent. For, the ratios of the terms in successive pairs in **(A)** are $\frac{x^2}{2.3}$, $\frac{x^2}{4.5}$, $\frac{x^2}{6.7}$, and so on, and in **(B)**, $\frac{x^2}{3.4}$, $\frac{x^2}{6.7}$, $\frac{x^2}{9.10}$, and on on. Whatever the value of x we may continue these fractions until the denominator is, say, greater than $3x^2$. Then the sum of the remaining terms in each series will be less than the sum of a geometric progression of which the ratio is $\frac{1}{3}$; hence (Alg. Ch. IX) each series is convergent. The convergence may also be shown for all ordinary values of x by (Alg. 500). On account of the rapid convergence, these series give a shorter method of computing a table of natural sines and cosines than that indicated in Chapter I.*

53. From **(A)** and **(B)** we may deduce, by the relations of (9)

$$\tan x = x+\frac{x^3}{3}+\frac{2x^5}{3.5}+\frac{17x^7}{3^2.5.7}+\frac{62x^9}{3^2.5.7.9}+\ldots\ldots \quad \textbf{(C)}.$$

$$\cot x = \frac{1}{x}-\frac{x}{3}-\frac{x^3}{5.9}-\frac{2x^5}{5.7.9}-\frac{x^7}{7.25.27}-\ldots\ldots \quad \textbf{(D)}.$$

$$\sec x = 1+\frac{x^2}{2}+\frac{5x^4}{2^3.3}+\frac{61x^6}{2^4.3^2.5}+\frac{277x^8}{2^7.3^2.7}+\ldots\ldots \quad \textbf{(E)}.$$

$$\csc x = x^{-1}+\frac{x}{2.3}+\frac{7x^3}{2^3.3^2.5}+\frac{31x^5}{2^4.3^3.5.7}+\frac{127x^7}{2^7.3^3.5^2.7}+\ldots\ldots \quad \textbf{(F)}.$$

*Elegant methods of computing Tables of Natural Functions, and also methods of computing directly Tables of Logarithmic Functions, which would be out of place, however, in a work of this character, may be found in Callet's "Tables de Logarithmes," Paris, 1879.

54. The relations expressed in (**A**). . . . (**F**) might evidently be taken as definitions, replacing those given in (**4**). The trigonometric functions might then be treated precisely as any other algebraic functions involving infinite converging series. We proceed to the purely algebraic treatment of these functions, and to some important extensions of Trigonometry, viewed as a department of pure algebra.

55. The Exponential Series.—Let us first expand a^x into a series arranged according to the ascending powers of x. Assume

$$a^x = P + Qx + Rx^2 + Sx^3 + Tx^4 + Vx^5 + \ldots\ldots$$

This equation, being identical, must be true for every value of x. Making $x=0$ we have $a^0=P$; *i. e.*, $P=1$. Hence

$$a^x = 1 + Qx + Rx^2 + Sx^3 + Tx^4 + Vx^5 + \ldots\ldots \quad (1).$$

Squaring both members of (1) we obtain

$$a^{2x} = 1 + 2Qx + 2R + Q^2 \mid x^2 + 2S + 2QR \mid x^3 + R^2 + 2QS + T \mid x^4 + \ldots . \quad (2).$$

Again, since (1) is an identity, we may substitute $2x$ for x; so doing we obtain

$$a^{2x} = 1 + 2Qx + 4Rx^2 + 8Sx^3 + 16Tx^4 + 32Vx^5 + \ldots \quad (3).$$

Equating the second members of (2) and (3), we obtain the identity

$$1 + 2Qx + 2R + Q^2 \mid x^2 + 2S + 2QR \mid x^3 + R^2 + 2QS + T \mid x^4 + \ldots . =$$
$$1 + 2Qx + 4Rx^2 + 8Sx^3 + 16Tx^4 + \ldots \quad (4).$$

Equating coefficients in (4), we obtain,

$$2R + Q^2 = 4R, \text{ whence } R = \frac{Q^2}{2};$$

$$2S + 2QR = 8S, \text{ whence } S = \frac{Q^3}{6} = \frac{Q^3}{3!};$$

$$R^2 + 2QS + T = 16T, \text{ whence } T = \frac{Q^4}{24} = \frac{Q^4}{4!};$$

etc., etc.

Substituting the values thus found in (1), we obtain,

$$a^x = 1 + Qx + \frac{Q^2}{2}x^2 + \frac{Q^3}{3!}x^3 + \frac{Q^4}{4!}x^4 + \ldots\ldots \quad (5),$$

the required series, in which Q, independent of x, depends only on the value of a. Since x may have any value whatever, let $x = 1 : Q$. Then

$$a^x = a^{1:Q} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots\ldots \quad (6).$$

The second member of (6) is e, the **Natural Base,** (Alg. 511); hence $a^{1:Q} = e$, or $e^Q = a$; i. e., $Q = \ln a$, so that

$$a^x = 1 + (\ln a)\,x + (\ln a)^2\frac{x^2}{2} + (\ln a)^3\frac{x^3}{3!} + (\ln a)^4\frac{x^4}{4!} + \ldots\ldots \quad (7).$$

56. In (7), (**55**), let $a = e$; we then have

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\ldots\ldots \quad (1).$$

As (1) is an identity, $x\sqrt{-1}$ may be substituted for x. Making this substitution, denoting $\sqrt{-1}$ by i, we obtain

$$e^{xi} = 1 + xi - \frac{x^2}{2} - \frac{x^3 i}{4!} + \frac{x^4}{3!} + \frac{x^5 i}{5!} - \frac{x^6}{6!} - \ldots\ldots \quad (2).$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

$$+ i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots\right),$$

or, by (**52**),

$$e^{xi} = \cos x + i \sin x \quad (3).$$

Similarly we may show that

$$e^{-xi} = \cos x - i \sin x \quad (4).$$

Combining (3) and (4) we obtain,

$$e^{xi} + e^{-xi} = 2\cos x;\ \therefore \cos x = \tfrac{1}{2}(e^{xi} + e^{-xi}) \quad (5).$$

$$e^{xi} - e^{-xi} = 2i \sin x;\ \therefore \sin x = \tfrac{1}{2i}(e^{xi} - e^{-xi}) \quad (6).$$

These relations may also be expressed thus:—

$$e^{xi} + e^{-xi} = \cos x + \cos(-x). \quad (7).$$

$$e^{xi} - e^{-xi} = i \sin x - i \sin(-x). \quad (8).$$

(7) and (8) are called **"Euler's Equations."**

57. The trigonometric functions are, then, merely **expo-**

nential functions of the arc, and may, therefore, be thus defined :—

$$\sin x = \frac{1}{2i}(e^{xi} - e^{-xi}) \quad (1); \quad \tan x = \frac{e^{xi} - e^{-xi}}{i\,(e^{xi} + e^{-xi})} \quad (3);$$

$$\cos x = \frac{1}{2}(e^{xi} + e^{-xi}) \quad (2); \quad \cot x = \frac{i\,(e^{xi} + e^{-xi})}{e^{xi} - e^{-xi}} \quad (4);$$

$$\sec x = \frac{2}{e^{xi} + e^{-xi}} \quad (5); \quad \csc x = \frac{2i}{e^{xi} - e^{-xi}} \quad (6).$$

All the formulæ established with the old definitions may now be established analytically with these, so that Trigonometry may be considered merely a department of Algebra.

58. Fundamental Relations.—Multiplying (3) and (4) (**56**), we obtain

$$1 = \cos^2 x + \sin^2 x.$$

This may also be obtained by direct substitution. Again, squaring the fractions defined as $\tan x$ and $\sec x$, and subtracting the first square from the second, we obtain

$$\sec^2 x - \tan^2 x = 1. \qquad \text{Similarly,}$$

$$\csc^2 x - \cot^2 x = 1.$$

The other relations are mere restatements of definitions.

59. Limiting Values.—By substituting, successively, $\frac{1}{2}\pi$, π, $\frac{3}{2}\pi$, 2π, etc., for x, in (1) . . . (6), (**57**), we find the same set of limiting values as given in the Table on page 14.

60. Again, by substituting for x in (5), (6), (**56**), $\frac{1}{2}\pi - x$, remembering that, by equations (3), (4), when $x = \frac{1}{2}\pi$, $e^{xi} = i$, and $e^{-xi} = -i$, we shall obtain,

$\sin x = \cos(\frac{1}{2}\pi - x)$; and similarly,

$\tan x = \cot(\frac{1}{2}\pi - x)$, and $\csc x = \sec(\frac{1}{2}\pi - x)$, as in (**5**).

61. The addition and subtraction formulæ of Chapter III may be established by direct substitution of the values of $\sin x$, $\cos x$, etc., as given in the definitions of (**57**). Or, we may proceed thus :—

$$e^{(x+y)i} = \cos(x+y) + i\sin(x+y) \qquad (a), \text{ by (3) (56).}$$

But

$$e^{xi+yi} = e^{xi}e^{yi} = (\cos x + i\sin x)(\cos y + i\sin y) \qquad (b).$$

Equating the values (a) and (b) we obtain

$(\cos x + i \sin x)(\cos y + i \sin y) = \cos(x+y) + i \sin(x+y)$,

or, expanding in the first member,

$$\cos x \cos y - \sin x \sin y + i(\sin x \cos y + \cos x \sin y) = \cos(x+y) + i \sin(x+y).$$

As the real parts of the two members of the identity must be equal,

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad (1).$$

As the coefficients of i in the two members must be equal,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad (2).$$

Since y may be negative in (1) and (2), the formulæ for $\cos(x-y)$ and $\sin(x-y)$ are established. From these we derive at once the formulæ for $\tan(x \pm y)$, $\cot(x \pm y)$, and the remaining formulæ of Chapter III.

62. Demoivre's Theorem.—In (3), (4), of (**56**), substitute nx for x; there results

$$e^{nxi} = \cos nx + i \sin nx\,; \quad e^{-nxi} = \cos nx - i \sin nx.$$

As $e^{nxi} = (e^{xi})^n$, and $e^{-nxi} = (e^{-xi})^n$, we obtain, on substituting for e^{xi} and e^{-xi} their values, (**56**),

$(\cos x \pm i \sin x)^n = \cos nx \pm i \sin nx$, *Demoivre's Theorem.*

63. It is clear that in the functions we have been considering x is not necessarily real. On the other hand it is plain that we may define and discuss a series of exponential functions, analogous to those already treated, but having the exponents real. From these we may develop a theory as complete and interesting as that of the ordinary trigonometric functions. These latter have those relations to the circular arc of length x, which we have already discussed, (**11**), whence they are usually called **circular** functions. The functions, the treatment of which we are about to undertake, have analogous relations to the arc of the equiangular hyperbola, whence they are called **hyperbolic** functions. Taking again the natural base, e, we define as follows:—

$$\textit{the hyperbolic sine of } x,\ (\sinh x) = \frac{1}{2}(e^x - e^{-x}) \qquad (1);$$

$$\textit{the hyperbolic cosine of } x,\ (\cosh x) = \frac{1}{2}(e^x + e^{-x}) \quad (2);$$

$$\textit{the hyperbolic tangent of } x,\ (\tanh x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad (3);$$

$$\textit{the hyperbolic cotangent of } x,\ (\coth x) = \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad (4);$$

$$\textit{the hyperbolic secant of } x,\ (\operatorname{sech} x) = \frac{2}{e^x + e^{-x}} \quad (5);$$

$$\textit{the hyperbolic cosecant of } x,\ (\operatorname{csch} x) = \frac{2}{e^x - e^{-x}} \quad (6).$$

From these definitions we derive at once series and relations analogous to those of (**53, 55**). Thus,

$$e^x = \cosh x + \sinh x \ldots\ldots \quad (7);$$

$$e^{-x} = \cosh x - \sinh x \ldots\ldots \quad (8).$$

Whence, by using the exponential series, we deduce

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots\ldots \quad (9);$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots\ldots \quad (10):$$

and from these we may derive, by mere division, series for $\tanh x$, $\coth x$, $\operatorname{sech} x$ and $\operatorname{csch} x$. The convergence of these series may be shown as in the case of the analogous series for the circular functions.

64. Fundamental Relations.—These, for the hyperbolic functions, are :—

$$\cosh^2 x - \sinh^2 x = 1 \quad (1); \qquad \sinh x \operatorname{csch} x = 1 \quad (5);$$

$$\tanh x = \frac{\sinh x}{\cosh x} \quad (2); \qquad \cosh x \operatorname{sech} x = 1 \quad (6);$$

$$\coth x = \frac{\cosh x}{\sinh x} \quad (3); \qquad \tanh^2 x + \operatorname{sech}^2 x = 1 \quad (7);$$

$$\tanh x \,.\, \coth x = 1 \quad (4); \qquad \coth^2 x - \operatorname{csch}^2 x = 1 \quad (8).$$

(2), (3), (4), (5), (6) are evident from equations (1)–(6), (**63**). To establish (1) we may multiply (7) by (8), (**63**), member by member. To establish (7) and (8) we may substitute at once the exponential values of $\tanh x$, etc., from (1)–(6) (**63**).

65. Limiting Values.—*Sinh x.* When $x = 0$ $\sinh x = \frac{1}{2}(1 - 1) = 0$. As x increases $\sinh x$ rapidly increases, becoming infinite when $x = \infty$. If x, diminishing from 0, becomes negative, $\sinh x$ becomes negative, and numerically increases, until, when $x = -\infty$, $\sinh x = -\infty$. It will be observed that, as in the case of $\sin x$, changing the sign of x changes the sign of $\sinh x$.

Cosh x. When $x = 0$, $\cosh x = 1$, and this is the smallest value of $\cosh x$; as x increases $\cosh x$ increases very rapidly, until, when $x = \infty$, $\cosh x = \infty$. Since, however,

$$\cosh^2 x - \sinh^2 x = 1,$$

$\cosh x$ always exceeds $\sinh x$. Again, $\cosh x = \cosh(-x)$; *i. e.*, changing the sign of x does not affect the sign of $\cosh x$.

Tanh x. Tanh $0 = 0$, and as x increases $\tanh x$ increases, from 0 onward, but very slowly. As x approaches ∞, $\tanh x$ approaches 1; when $x = \infty$, $\tanh x = 1$. For,

$$\frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1}{1 + e^{-2x}} - \frac{e^{-2x}}{1 + e^{-2x}} = \frac{1}{1 + e^{-2x}} - \frac{1}{1 + e^{2x}}$$

$= 1$, when $x = \infty$.

If x, passing through 0, becomes negative, $\tanh x$ becomes negative, and passes through the same numerical changes as in the case of positive x.

Coth x. When $x = 0$, $\coth x = 1 : 0 = \infty$; as x increases $\coth x$ decreases, but remains the reciprocal of $\tanh x$, and, when $x = \infty$, $\coth x = 1 : 1 = 1$.

For negative arcs the numerical changes in the $\coth x$ are the same as in the case of positive arcs.

Sech x and *Csch x.* These functions are the reciprocals respectively of the $\cosh x$ and $\sinh x$. Hence, as x passes from 0 to ∞, $\operatorname{sech} x$ passes from 1 to 0, and as x passes from 0 to $-\infty$, $\operatorname{sech} x$ again passes from 1 to 0. As x passes from 0 to ∞, $\operatorname{csch} x$ passes from ∞ to 0, and as x passes from 0 to $-\infty$, $\operatorname{csch} x$ passes from $-\infty$ to 0.

66. Addition and Subtraction Formulæ.—By direct

substitution of the values (**63**) of the functions we may readily show that:—

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \quad (1);$$
$$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y \quad (2);$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y \quad (3);$$
$$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y \quad (4);$$
$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \quad (5);$$
$$\coth(x \pm y) = \frac{\coth x \coth y \pm 1}{\coth y \pm \coth x} \quad (6).$$

We may also derive these from the corresponding formulæ for circular functions (and *vice versa*), by using the following relations, which are evident from the definitions, viz.:—

$$\sin x = \frac{1}{i}\sinh ix \ (1); \quad \tan x = \frac{1}{i}\tanh ix \ (3);$$
$$\cos x = \cosh ix \ (2); \quad \cot x = i \coth ix \ (4).$$

67. Periodicity of the Functions.—The circular functions are *periodic; i. e.*, as x passes from 0 to ∞ the same values of the functions recur at intervals of 2π. The functions of x are the same as those of $(x+2n\pi)$, n being any integer. The values of $\cos x$, $\sin x$, $\sec x$, $\csc x$, recur at intervals of 2π; the values of $\tan x$ and $\cot x$ at intervals of π. The relations just expressed indicate a similar periodicity in the hyperbolic functions. But while the circular functions are periodic for real values of x, and continuous for imaginary values, the hyperbolic functions, on the other hand, are continuous for real values of x (**65**), and periodic for imaginary values. As

$$\cosh 2n\pi i = \cos 2n\pi = 1, \text{ and } \sinh 2n\pi i = i \sin 2n\pi = 0,$$

we have the following formulæ, analogous in pairs:—

CIRCULAR FUNCTIONS.	HYPERBOLIC FUNCTIONS.
$\sin(x+2n\pi) = \sin x;$	$\sinh(x+2n\pi i) = \sinh x;$
$\cos(x+2n\pi) = \cos x;$	$\cosh(x+2n\pi i) = \cosh x;$
$\tan(x+n\pi) = \tan x;$	$\tanh(x+n\pi i) = \tanh x;$
$\cot(x+n\pi) = \cot x;$	$\coth(x+n\pi i) = \coth x;$

The student should substitute, successively, $\frac{1}{2}\pi i$, πi, $\frac{3}{2}\pi i$, and $2\pi i$ for y in *all* the formulæ of (66), and thus construct a table for hyperbolic functions analogous to that on page 14.

68. By processes analogous to those of (**27, 28**) we may deduce expressions for the hyperbolic functions of $2x$ and of $\frac{1}{2}x$, and also for $\cosh a \pm \cosh b$, $\sinh a \pm \sinh b$, etc. Or, we may derive these formulæ directly from the definitions. Thus, as

$$(e^{\frac{1}{2}a+\frac{1}{2}b} + e^{-\frac{1}{2}a-\frac{1}{2}b})(e^{\frac{1}{2}a-\frac{1}{2}b} + e^{-\frac{1}{2}a+\frac{1}{2}b}) = e^a + e^{-a} + e^b + e^{-b},$$

we have

$$\cosh a + \cosh b = 2 \cosh \tfrac{1}{2}(a+b) \cosh \tfrac{1}{2}(a-b).$$

Similarly for the other formulæ.

69. The exponential character of the hyperbolic and circular functions suggests important relations that obtain between them and the ordinary logarithmic functions. For example, let $\cosh x = z$, so that $\sinh x = \sqrt{z^2-1}$, then, ((7), (8), 63),

$$x = \ln (z + \sqrt{z^2-1}), \text{ and } -x = \ln (z - \sqrt{z^2-1}).$$

Similarly, if $z = \sinh x$,

$$x = \ln (z + \sqrt{z^2+1}), \text{ and } -x = \ln (\sqrt{z^2+1} - z).$$

Again, since $\dfrac{1+\tanh x}{1-\tanh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$, (63); if $z = \tanh x$,

$$2x = \ln \frac{1+z}{1-z}, \text{ or } x = \frac{1}{2} \ln \frac{1+z}{1-z} = \ln \sqrt{\frac{1+z}{1-z}}.$$

$$\text{Further, if } w = \coth x, \quad x = \ln \sqrt{\frac{1+w}{1-w}}.$$

The corresponding relations for the circular functions are evidently,

$$xi = \ln (\sqrt{1-z^2} + zi) = -\ln (\sqrt{1-z^2} - zi) \qquad (1);$$

$$xi = \ln \sqrt{\frac{1+zi}{1-zi}} \qquad (2); \quad xi = \ln \sqrt{\frac{zi+1}{zi-1}} \qquad (3).$$

70. This connection of the Trigonometric functions with logarithms indicates that we are not confined to the *natural base*, e, in our definitions. Thus, if a be the base, instead of e,

since $a^x = e^{x \ln a}$, if, in all discussions we replace a by e, the only change required in the functions will be the change of the exponent from x to $x \ln a$, which will evidently not affect the nature or the results of our investigations.

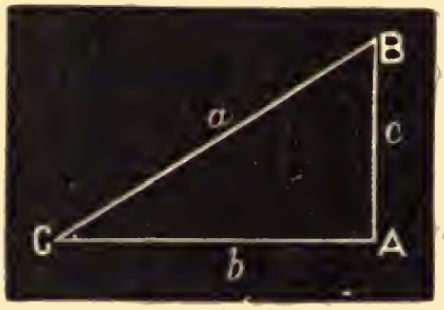


Fig. 23.

71. Geometric Interpretation of the Hyperbolic Functions.—In (4) we dealt with the relations of the circular functions to the sides of *triangles* of reference. Let us now investigate the analogous relations of the hyperbolic functions. If BAC, (Fig. 23), be right-angled at A, we have

$$\sin C = c : a;\ \cos C = b : a;\ \tan C = c : b.$$

Now let BA revolve about B, the angle C remaining constant, and let x be an arc, some function of the measuring arc of C, such that

$$\sinh x = c : a;\ \cosh x = b : a;\ \tanh x = c : b.$$

We are to determine the proper shape of the triangle, and the relation between C and x. Since, (22),

$$\frac{\sin C}{\sin A} = \frac{c}{a};\ \frac{\sin B}{\sin A} = \frac{b}{a};\ \frac{\sin C}{\sin B} = \frac{c}{b};$$

and since $\cosh^2 x - \sinh^2 x = 1$ (α); we obtain, on substituting in (α) for $\cosh x$ and $\sinh x$ their values,

$$\sin^2 B - \sin^2 C = \sin^2 A = \sin^2 (C + A) - \sin^2 C;$$

or, since $\sin^2 x - \sin^2 y = \sin (x + y) \sin (x + y)$

$$\sin (2C + A) \sin A = \sin^2 A;\ \text{whence}$$

$$\sin (2C + A) = \sin A\ (1),\ \text{or}\ \sin A = 0\ (2).$$

The latter value of sin A being inadmissible, we have, from (1),

$$2C + A = \pi - A,\ \text{or}\ C + A = \tfrac{1}{2}\pi;$$

$$\text{whence}\ B = \tfrac{1}{2}\pi;\ \textit{i. e.},\ BA \perp CB.$$

This determines the *shape* of the triangle. The relation between C and x is evident; for

$$\sinh x = \tan C;\ \tanh x = \sin C;$$

$$\cosh x = \sec C;\ \coth x = \csc C.$$

72. If, in any circle, (Fig. 24), a perpendicular, PM, be dropped from the termination, (M), of any arc on the diame-

ter through the origin thereof, and if PM $= y$, OM x, and radius $= a$, then evidently,

$$x^2 + y^2 = a^2.$$

Hence we may at once write:—

$x = a \cos \theta$; $y = a \sin \theta$, if θ denote the radial measure of AOP. This convention gives us the circular functions as defined geometrically in Chapter I. $\frac{1}{2}a^2\theta$ is the area of the sector AOP, and hence, if $a = 1$, θ represents double this area.

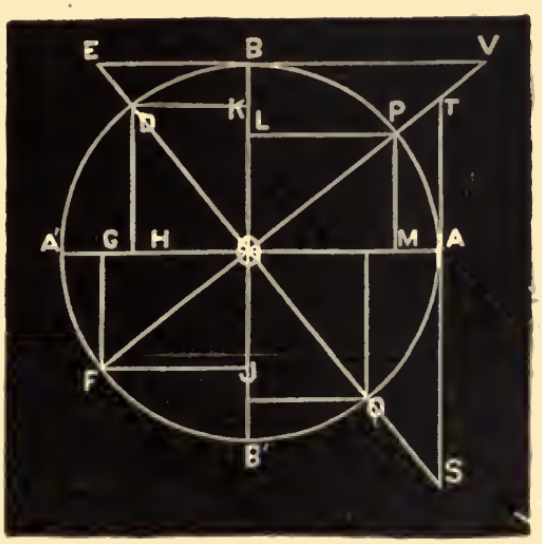


Fig. 24.

The hyperbolic functions are connected in an analogous way with the rectangular hyperbola. Thus, (Fig. 25), if PM be the perpendicular dropped from any point P, of a rectangular hyperbola on the diameter through the origin, (A), of the hyperbolic arc, and if, as before, PM $= y$, OM $= x$, and the radius of the auxiliary circle be a. we have. from the well-known property of the curve,

$$x^2 - y^2 = a^2.$$

Hence we may define the hyperbolic functions geometrically as follows:—

(1) The *hyperbolic sine* is the perpendicular dropped from the termination of the hyperbolic arc on the diameter through the origin, divided by the radius of the auxiliary circle. This radius is to the hyperbola what the radius is to the circle, viz., the semi-axis of the curve.

(2) The hyperbolic cosine is the distance from the centre to the foot of the hyperbolic sine (the base of the hyperbolic triangle of reference), divided by the semi-axis (a).

In a circle every diameter is an axis; hence any semi-diameter (or radius) may be taken as a semi-axis. In the hyperbola, however, this is not the fact; hence we take the only semi-diameter that is also a semi-axis, as the denominator of the hyperbolic functions.

The definitions of the geometric representatives of the other hyperbolic functions are at once evident (**63**).

73. As will be shown later, the area of the hyperbolic sector AOP is $\frac{1}{2}a^2u$ (u being the hyperbolic arc AP), and hence, just as in the case of the circle, if $a = 1$, u is double the area of AOP.

From M draw ML, tangent to the auxiliary circle (of radius a), at L, and denote the angle (arc) AOL by θ. Then

$$a \sec\theta = \text{OM} = a\cosh u;$$
$$a\tan\theta = \text{LM} = a\sinh u = \text{PM}.$$

Fig. 25.

When θ is thus connected with u it is called the Gudermannian of u,* ordinarily abbreviated gd u. A triangle of reference for any hyperbolic function of u may be readily constructed; the calculation of any hyperbolic function may thus be reduced to the calculation of a circular function. For example, lay off

MK $= a$, and draw PK; tan MKP $=$ PM $: a = \sinh u$.

Similarly for the other functions.

74. Hyperbolic and circular functions are of great importance in the Integral Calculus, where it will be shown that they are special forms of certain more general functions, known as Elliptic Functions. Any attempt at a discussion of these latter would, however, be entirely beyond the scope of this work.

Interesting examples of the methods of employing these functions in the solution of equations will be found in Todhunter's and Burnside and Panton's treatises on the Theory of Equations, and also in the various mathematical journals, among which Crelle's is perhaps the best. These discussions are not, however, deemed of sufficient practical importance to warrant their introduction into the present work.

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*cf. Crelle, "Journal für die reine und angewandte Mathematik," Band 6.

EXERCISES.

CHAPTER I.

1. Construct

$\sin^{-1}(.3)$; $\tan^{-1}(6)$; $\cot^{-1}(-4)$; $\sec^{-1}(-3)$; $\cos^{-1}(-.6)$;

$\sin^{-1}\left(\frac{\pi}{4}\right)$; $\tan^{-1}\pi$; $\csc^{-1}\left(-\frac{\pi}{2}\right)$; $\cos^{-1}\left(\frac{1}{\pi}\right)$.

Determine (with a protractor) these angles, to within 1°; determine also their radial measure.

2. Determine, with protractor and scale, the functions of 5°, 10°, 15°, etc., to 180° inclusive.

3. Determine geometrically the functions of $(90° \pm x)$, $(180° \pm x)$, $(270° \pm x)$, and $(360° \pm x)$, in terms of functions of x.

4. Find the functions of

$$\sin^{-1}\frac{3}{5};\quad \cos^{-1}\frac{a}{b};\ \tan^{-1}c;\ \cot^{-1}(-2);\ \csc^{-1}(-m).$$

5. Express successively all the functions of x in terms of each.

6. State the signs of the functions of

$$(2n+1)180° \pm \mathrm{A};\quad 2n\pi \pm \mathrm{A};\quad (2n-1)\pi - \frac{6n+1}{6}\pi.$$

7. Prove that

(1) $$\frac{\sec x + \csc x}{\sec x - \csc x} = \frac{1+\cot x}{1-\cot x} = \frac{\tan x + 1}{\tan x - 1}.$$

(2) $$\frac{\sec x + \csc x - \sec x \csc x}{\sec x \csc x} = 2\,\frac{\sin x \cos x \sec x \csc x}{\sec x + \csc x + \sec x \csc x}.$$

(3) $$\sqrt{\frac{1+\tan^2 x}{1+\cot^2 x}} = \tan x;\quad \tan x + \cot x = -\sec x \csc x.$$

(4) $\dfrac{\sec^2 x - 1}{\csc^2 x - 1} = \dfrac{1 + \cos x}{1 + \sin x} \cdot \dfrac{\operatorname{versin} x}{\operatorname{coversin} x} \cdot \tan^2 x.$

(5) $\tan^{-1}\dfrac{x}{x+1} + \cot^{-1}(2x+1) = \dfrac{\pi}{4}.$

(6) $\tan A + \cot A = \sec A \csc A.$

(7) $\sec B + \sqrt{\sec^2 B - 1} = \dfrac{1}{\sec B - \sqrt{\sec^2 B - 1}}.$

(8) $\dfrac{1 - \sin A}{1 + \sin A} = 1 + 2\tan^2 A - 2 \sec A \tan A.$

(9) $\dfrac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B.$

(10) $\sec^2 A \csc^2 A - (\sec^2 A + \csc^2 A) = 0.$

(11) $\sin A + \cos A = \sqrt{\dfrac{\sec A \csc A + 2}{\sec A \csc A}}.$

8. Solve, analytically and graphically, the following equations:—

(1) $2\cos^2 x = \tan x$; (2) $\csc x = 4 \sin x$; (3) $\sin x = 2 \cos x.$

(4) $\sin x + \cos x = \csc x$; (5) $5 \sec^2 x - 12 \sin^2 x = 4 \tan x.$

(6) $\sin x \cos x + \tan x \cot x = \dfrac{3}{\sec x \csc x}.$

(7) $2(\sin x + \cos x) + \sec x \csc x - \cot x = 3.$

(8) $3 \sec x \csc x \tan x = \cot^2 x + 1.$

(9) $3 \sin x + 5 \cos x = 8$; (10) $2 \tan x + 3 \csc x = 6.$

(11) $\cot x - 4 \sin x = 7$; (12) $\sin^2 x + 7 \cos x = 1.$

(13) $3 \cos^4 x + 6 \sin^2 x \cos x = 1.$

(14) $\sec x - \tan x = 4\sqrt{\sin x} - 2.$

(15) $a \cos x + b \sin x = c.$

9. Show that, for any angle,

$$2(\sin^4 A + \cos^4 A + 2 \sin^2 A \cos^2 A) =$$
$$(\tan A + \cot A)^2 - \tan^2 A - \cot^2 A = \text{constant};$$
$$\frac{\cos A + \cos B}{\sin A - \sin B} = \frac{\sin A + \sin B}{\cos B - \cos A}.$$

10. Solve the equations $r \sin x = a$ (1)

$r \cos x = b$ (2)

CHAPTER II.

1. Solve the right triangle ABC, given

(1) $b = 3$; $c = 4$; (4) $a = 625$ $B = 52° 13'$;

(2) $a = 29$; $b = 20$; (5) $b = .00216$ $B = 87° 26'$;

(3) $a = 30.126$; $C = 27° 50'$; (6) $c = .00326$ $C = 2° 15'$.

2. When the sun's altitude is $36° 19'$ the shadow of a flag-pole is 175 ft. What is the height of the pole? What is the length of the shadow when the sun's altitude increases to $40° 16' 27''$?

3. A winding road 3 miles long has a rise of 1,196 feet. Required the angle of (uniform) slope.

4. The angles of elevation of the top of an inaccessible crag are $32° 46'$, and $24° 17'$, as observed from two points, 200 yards apart, in the same vertical plane with the top. If the theodolite stands four feet from the ground, what is the height of the crag?

5. Two towers stand on opposite banks of a river. From the top of one, 168 feet high, the angles of depression of the top and bottom of the other are observed to be $13° 28'$, and $28° 16'$, respectively. Required the breadth of the river and the height of the second tower.

6. What is the angle of depression of a tangent to the earth's surface as observed from the top of a mountain 18,260 feet high? What is the distance from the foot of the mountain at which the top must cease to be visible, the average slope being $28° 17'$?

Fig. 26.

7. Let E represent the earth and M the moon. The angle EMS $= 57'$, (moon's parallax); the radius of the earth is 3,956 miles. Required the geocentric distance of the moon.

8. The angle MET (the moon's apparent semi-diameter), is $31' 20''$ What is the diameter of the moon?

9. To find the point at which a perpendicular from an inaccessible point will meet a given line.

10. In a right triangle are given R and r. To solve.

11. Find the volume of a cone of revolution whose vertical angle is 75° 16′ 23″, and whose slant height is 20.136.

12. To solve an isosceles triangle, when the base and r are given.

13. Each side of the base of a regular heptagonal pyramid is 10.307, each lateral edge makes with the base an angle of 39° 46′ 19″. Required the volume and surface of the pyramid.

14. To solve a right triangle, given R and B, or R and b.

CHAPTER III.

1. Establish geometrically the formulæ of (**27–8–9**).

2. Show that:—

(1) $\sin(30°+A)+\sin(30°-A)=\cos(60°+A)\cos(60°-A)$
$=\cos A$;

(2) $\sec\frac{1}{2}A=\dfrac{\cot\frac{1}{2}A+\tan A}{\cot\frac{1}{2}A}$; $\sec A=\dfrac{\cot\frac{1}{2}A+\tan\frac{1}{2}A}{\cot\frac{1}{2}A-\tan\frac{1}{2}A}$;

(3) $\sin 3x=\sin x\,(4\cos^2x-1)=3\sin x-4\sin^3x$,
$\sin 4x=\sin x\,(\ 8\cos^3x-\ 4\cos x)$,
$\sin 5x=\sin x\,(16\cos^4x-12\cos^2x+1)$,
$\sin 6x=\sin x\,(32\cos^5x-32\cos^3x+6\cos x)$,
$\sin 7x=\sin x\,(64\cos^6x-80\cos^4x+24\cos^2x-1)$,
$\sin 8x=\sin x\,(128\cos^7x-192\cos^5x+80\cos^3x-8\cos x)$;
. . . . . .

(4) $\cos 3x=\ 4\cos^3x-\ 3\cos x$,
$\cos 4x=\ 8\cos^4x-\ 8\cos^2x+\ 1$,
$\cos 5x=16\cos^5x-20\cos^3x+\ 5\cos x$,
$\cos 6x=32\cos^6x-48\cos^4x+18\cos^2x-\ 1$,
$\cos 7x=64\cos^7x-112\cos^5x+56\cos^3x-\ 7\cos x$,
$\cos 8x=128\cos^8x-256\cos^6x+160\cos^4x-32\cos^2x+1$.

Establish also the formulæ for the tangents and cotangents of multiples of x.

$$\cos^2 k = \frac{1}{2}\cos 2k + \frac{1}{2},$$

$$\cos^3 k = \frac{1}{4}\cos 3k + \frac{3}{4}\cos k,$$

$$\cos^4 k = \frac{1}{8}\cos 4k + \frac{1}{2}\cos 2k + \frac{3}{8},$$

$$\cos^5 k = \frac{1}{16}\cos 5k + \frac{5}{16}\cos 3k + \frac{5}{8}\cos k,$$

$$\cos^6 k = \frac{1}{32}\cos 6k + \frac{3}{16}\cos 4k + \frac{15}{12}\cos 2k + \frac{5}{16},$$

$$\cos^7 k = \frac{1}{64}\cos 7k + \frac{7}{64}\cos 5k + \frac{21}{64}\cos 3k + \frac{35}{64}\cos k,$$

$$\cos^8 k = \frac{1}{128}\cos 8k + \frac{1}{16}\cos 6k + \frac{7}{32}\cos 4k + \frac{7}{16}\cos 2k + \frac{35}{128}.$$

3. Show that:—

$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y} \quad (1); \qquad \tan x \pm \cot y = \frac{\cos(x \mp y)}{\cos x \sin y} \quad (3);$$

$$\cot x \pm \cot y = \frac{\sin(y \pm x)}{\sin x \sin y} \quad (2); \qquad \cos(x \pm y) = \frac{1 \mp \tan x \tan y}{\sec x \sec y} \quad (4).$$

What do (1), (2), (3), become when

$$x = a - \tfrac{1}{4}\pi, \text{ and } y = a + \tfrac{1}{4}\pi?$$

(5) $\sin(x+y)\sin(x-y) = \sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$;

(6) $\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y = \cos^2 y - \sin^2 x$.

From (6) deduce the value of $\cos 2x$.

(7) $(\tan^2 x - \tan^2 y)(\cot^2 x - \cot^2 y) = ?$

4. Show that $\tan \tfrac{1}{2}x = \dfrac{1 - \cos x}{\sin x}$, and $\cot \tfrac{1}{2}x = \dfrac{1 + \cos x}{\sin x}$

Prove that $\tan 2x = \dfrac{\sin x + \sin 3x}{\cos x + \cos 3x}$; also that

$$\tan 3x = \frac{\sin 2x + \sin 4x}{\cos 2x + \cos 4x};$$

$$\tan 4x = \frac{\sin x + \sin 3x + \sin 5x + \sin 7x}{\cos x + \cos 3x + \cos 5x + \cos 7x};$$

$$\tan 5x = \frac{\sin 2x + \sin 4x + \sin 6x + \sin 8x}{\cos 2x + \cos 4x + \cos 6x + \cos 8x}.$$

Deduce the corresponding expression for $\tan nx$.

Show that $\cot x = \tan x + 2\tan 2x + 4\tan 4x + 8\tan 8x$.

5. Prove that:—

(1) $\cos(x+y) \pm \sin(x-y) = \sqrt{2}\sin(\frac{1}{4}\pi \pm x)\cos(\frac{1}{4}\pi \pm y)$;

(2) $\cos y = \cos(x+y)\cos x + \sin(x+y)\sin x$;

(3) $\tan^{-1}\frac{x}{x+1} + \cot^{-1}(2x+1) = \frac{1}{4}\pi$;

(4) $2\cot^{-1}3 + \cot^{-1}7 = \frac{1}{4}\pi$;

(5) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}\frac{1-x-y-xy}{1+x+y+xy} = \frac{\pi}{4}$.

6. Given $\sin 2x = \cos 3x$; find x and its functions.

7. From the functions of 30°, 45°, and 18°, find the functions of all those angles between 0° and 90°, the expressions for which involve no fraction less than 0.°5, and the calculation of whose functions requires no other data.

8. Show that:—

(1) $\sin 2x = \frac{2\tan x}{1+\tan^2 x} = \frac{2}{\tan x + \cot x} = 2 - 2\frac{\cos^3 x + \sin^3 x}{\cos x + \sin x}$;

(2) $\cos 2x = \frac{\cot^2 x - 1}{\cot^2 x + 1} = (1 \pm \sin 2x)\frac{\cot x \mp 1}{\cot x \pm 1}$;

(3) $\cot\frac{x}{2} = \frac{1+\sin x+\cos x}{1+\sin x-\cos x}$.

9. Express, as the sum or difference of two functions, (1), $2\sin 3x\cos 2y$; (2) $\sin 3x\cos 5x$; (3) $\sin 20°\sin 30°$.

Simplify $\frac{\sin 3x \pm \sin 5x}{\cos 3x \pm \cos 5x}$; $\frac{\cos 4x \pm \cos 6x}{\sin 4x \mp \sin 6x}$.

Prove that $\tan 5x - \tan 3x - \tan 2x = \tan 5x\ \tan 3x\ \tan 2x$.

10. Solve the following equations:—

(1) $\cos x - \sin x = (\cos x + \sin x)(2\cos^2 x - 1)$;

(2) $\cos(x + \frac{1}{3}\pi) + 2\sin x + 3\tan(x + \frac{1}{12}\pi) = \cot^2 x + \csc x$;

(3) $2\tan x + \tan(a-x) = \tan(b+x)$;

(4). $2a \sec x = (a+b)(1 - \tan x + \sec x)$;

(5). $\sqrt{\sin x - \csc x} - \sqrt{1 - \csc x} = \csc x\,(\sin x - 1)$.

11. Show that:—

$$\frac{\sin A + \sin B + \sin (A+B)}{\sin A + \sin B - \sin (A+B)} = \cot \frac{A}{2} \cot \frac{B}{2} \qquad (1);$$

$$\tan 3A \cot A\,(2 \cos 2A - 1) = 2 \cos 2A + 1 \qquad (2).$$

12. Sum the series, $\sin x + a \sin 2x + a^2 \sin 3x + \ldots\ldots\ldots$

Assuming that $x = \cos m + \sqrt{-2} \sin m$, sum $x + x^2 + x^3 + \ldots$.

13. Show that, if $C = 90°$,

$$\frac{a+b}{a-b} + \frac{a-b}{a+b} = \frac{4 \sin^2 A}{\cos 2B} - 2.$$

14. $\cos (x+y+z) + \cos (x+y-z) + \cos (x-y+z) + \cos (z+y-x) = ?$

15. Show that if

$$\begin{vmatrix} 1 & \cos x & \cos y & 0 \\ \cos x & 1 & \cos z & 0 \\ \cos y & \cos z & 1 & \cos w \\ 0 & 0 & \cos w & 1 \end{vmatrix} = 0,$$

$\sin^2 x \cos^2 w = 1 - \cos^2 z - \cos^2 y - \cos^2 x + 2 \cos x \cos y \cos z$.

16. Evaluate:—

$$\begin{vmatrix} 1, & 1, & 1, \\ \sin x, & \sin y, & \sin z, \\ \cos x, & \cos y, & \cos z, \end{vmatrix}, \quad \begin{vmatrix} \sin x, & \sin y, & \sin z, \\ \cos x, & \cos y, & \cos z, \\ \sin 2x, & \sin 2y, & \sin 2z, \end{vmatrix}, \quad \begin{vmatrix} \sin x, & \sin y, & \sin z, \\ \sin 2x, & \sin 2y, & \sin 2z, \\ \sin 4x, & \sin 4y, & \sin 4z, \end{vmatrix}$$

17. Given, (1), $\sin x - \frac{1}{2} \sin 2x = m$;

(2), $\cos x - \frac{1}{2} \cos 2x = k$: find x.

18. Eliminate x and y from the groups of equations:—

$$r \cos x \sec y - (r : a) \csc (x+y)$$
$$= [(t : b) - \sin x \sec y] \sec (x+y) = \tan y \qquad (1),$$
$$r \cos (x - 3y) = 2a \cos^3 y; \quad r \sin (x - 3y) = 2a \sin^3 y \qquad (2).$$

19. Show that two geometrical progressions may be formed from the roots of

$$x^2 - 2x \cos a + 1 = 0,$$
$$x^2 - 2x \cos 2a + 1 = 0,$$
$$\ldots\ldots\ldots\ldots$$
$$x^2 - 2x \cos na + 1 = 0;$$

the product of the two series being $\sin^2\left(\frac{na}{2}\right)\operatorname{csc}^2\left(\frac{a}{2}\right)$.

20. Show that:—

$$\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix} = 0; \quad \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} = 0;$$

$$\begin{vmatrix} 0 & b & c \\ b & 1 & \cos A \\ c & \cos A & 1 \end{vmatrix} - \begin{vmatrix} a & b & c \\ b - c\cos A & a & 0 \\ c - b\cos A & 0 & a \end{vmatrix} = a^2.$$

21. Show that

$$\begin{vmatrix} \cos\frac{1}{2}(B-C) & \cos\frac{1}{2}(C-A) & \cos\frac{1}{2}(A-B) \\ \cos\frac{1}{2}(B+C) & \cos\frac{1}{2}(C+A) & \cos\frac{1}{2}(A+B) \\ \sin\frac{1}{2}(B+C) & \sin\frac{1}{2}(C+A) & \sin\frac{1}{2}(A+B) \end{vmatrix}$$

equals $\sin(B-C)+\sin(C-A)+\sin(A-B)$.

22. If A, B, C, be the angles of a triangle:—

$$\frac{\cos A}{\sin B\sin C}+\frac{\cos B}{\sin A\sin C}+\frac{\cos C}{\sin B\sin C}=2 \qquad (1);$$

$$\sin^2 C=\cos^2 B+\cos^2 A+2\cos A\cos B\cos C \qquad (2);$$

$$\sin A+\sin B+\sin C=(\text{semi-perimeter}):R \qquad (3);$$

$$c+b:a::c-b:c\cos B-b\cos C \qquad (4);$$

$$b\cos(X-C)+c\cos(X+B)=b\cos(X+C)+c\cos(X-B) \qquad (5);$$

$$bc\div 4R^2=\cos A+\cos B\cos C=2(\text{Area})\sin A\div a^2 \qquad (6);$$

$$2R\sin C=b\cos A+\sqrt{(a^2-b^2\sin^2 A)} \qquad (7).$$

23. What is the shape of a triangle in which

$$a^2+b^2:a^2-b^2=\sin(A+B):\sin(A-B)?$$

24. In any right triangle, $(A=90°)$:—

$$\sin 2B+\sin 2C=4ab\div c^2 \qquad (1);$$

$$\sin 2B-\sin 2C=\cos 2B+\cos 2C=0 \qquad (2);$$

$$\sin^2 C+\sin^2 B-(\sin^2\tfrac{1}{2}B+\sin^2\tfrac{1}{2}C)=(b+c)\div 2a \qquad (3).$$

Express (3) in terms of cosines.

25. In any triangle, ABC,

$$b-c:a::\sin\tfrac{1}{2}(B-C):\cos\tfrac{1}{2}A::\cos C-\cos B:1+\cos A;$$

$$b+c:a::\cos\tfrac{1}{2}(B-C):\sin\tfrac{1}{2}A::\cos C+\cos B:1-\cos A;$$

$$a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C;$$

$$(\cos A\div b\sin C)+(\cos B\div c\sin A)+(\cos C\div a\sin B)=$$
$$2(\sin A+2\sin B)\div(a+2b).$$

26. If i, j, k, be the altitudes of a triangle,

$a \sin A + b \sin B + c \sin C = 2(i \cos A + j \cos B + k \cos C)$ (1);

$\sin A = \sin (2A + B + C) = -\cos \frac{1}{2}(3A + B + C)$ (2);

$\tan (B - 2C) = -\tan (A + 2C)$ (3);

$\csc (\frac{1}{2}B - C) = \sec (\frac{1}{2}A + C)$; $\tan (A + 2B + 3C) = \tan (A - C)$ (4);

$\cot \frac{1}{2}(B + C) \cot \frac{1}{2}(C + A) + \cot \frac{1}{2}(C + A) \cot \frac{1}{2}(A + B)$
$+ \cot \frac{1}{2}(B + C) \cot \frac{1}{2}(C + A) = 1$ (5).

27. Extend to quadrilaterals (and to polygons in general), the relation $a^2 = b^2 + c^2 - 2bc \cos A$. Extend also the relation

$$a = b \cos C + c \cos B.$$

28. The roots of a cubic are the tangents of the angles of a triangle. Required the form of the cubic. Investigate also the cubic whose roots are sines and cosines.

EXERCISES IN SOLVING PLANE TRIANGLES.

1. To solve a triangle ABC, given:—

(1) $a = 237.062$, $b = 127.268$, $A = 27° 32' 16''$.

(2) $a = 156.82$, $b = 284.072$, $c = 192.007$.

2. To solve a triangle ABC, given:—

(1) a, b, $A \pm B$;

(2) A, a, $b^2 \pm c^2$;

(3) A, C, $a \pm b$;

(4) c, A, $a^2 \pm b^2$;

(5) The altitudes;

(6) The medians;

(7) The bisectrices of the angles;

(8) The sides of the Pedal triangle;

(9) The perimeter, an altitude and a side;

(10) The radius of the inscribed circle, $a \pm b$, and c;

(11) The radius of the inscribed circle, the radius of the circumscribed circle, and one altitude.

3. The distance between two buoys, A and B, being required, a base line, CD, of 300 yards is laid off, and the angles, CDA, $(= 28° 19')$, ADB $(= 62° 16')$, ACB $(= 57° 38')$, and BCD $(= 38° 52')$ are measured. Required, AB, and the angle CBD.

4. Deduce expressions for the side, apothem and area of each regular polygon inscribed in a circle whose radius is 10, up to and including, the dodecagon.

5. An aeronaut observes the angles of depression of two towers on opposite shores of a lake, and 12 miles apart, to be 47° 58′ and 33° 25′, respectively. What is his altitude?

6. An engine passes a station, C, that is 8 miles distant, and that bears S 32° 17′ W, at 10 A. M.; moving continuously, it passes B, N 68° 26′ W, and 5 miles away, at 10 : 20 A. M. If the road BC is straight, what is the average rate of speed of the engine?

7. Show that, (Fig. 7, 8), $CD = \frac{c \sin A \sin B}{\sin (A + B)}$; express the area of ABC in terms of c, A, B.

8. The area of the largest circle that can be cut from a triangular block, is 1312. The area of the block is 2018, and the angle at one corner is 57° 10′. Required the dimensions of the block.

9. If ABC, DEF, have $a = d$, $b = e$, and $C = 180° - F$, they are equivalent. Generalize this statement.

10. To determine the distance from D to an invisible, inaccessible point, B.

11. Show that in any triangle, ABC:—

$\cot \frac{1}{2}A + \cot \frac{1}{2}B + \cot \frac{1}{2}C = s \div r$;

$\sin A + \sin B + \sin C = s \div R = \sin A \sin B \sin C \cdot 2R \div r$;

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$;

$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$.

12. A line AD is drawn from A to BC. Compare the ratios DAC : DAB, and DB : DC.

13. To find the area of an inaccessible field, knowing a base line KL, and the angles subtended at K and L by the sides of the field.

17. Express the area of a quadrilateral in terms of the diagonals and their included angle.

18. Determine a trapezoid, knowing the sides a, b, c, d; a being parallel to c.

19. B is due south of A. An observer at C notes the respective intervals (6 seconds, 9 seconds), between the flashes of

the sunset guns at B and A, and the times at which the reports reach him. AB subtends to him an angle of 68° 29′. How far is A from B?

20. A circular arch, 98 feet in (curvilinear) length, rests on piers 16 feet high and 82 feet apart (from centre to centre). What is the height of the highest point of the arch?

21. Three circles, of radii 8, 12, and 20, respectively, are tangent, each to the other two. Required the perimeter and area of the enclosed space.

22. Find the distance in space described in one day, in consequence of the rotation of the earth, by the Great Equatorial at the Lick Observatory of the University of California, at Mt. Hamilton, (Lat. 37° 20′ 22″.51 N. Long. 121° 38′ 35″.75 W).

23. Find the distance between the orthocentre and the centre of the circumscribed circle, (of ABC), in terms of the radius of the latter and the angles of ABC.

24. Express, in terms of the radii, the distance between the centres of the inscribed and circumscribed circles. Express, (in terms of a, b, c), the distances between the centres of the escribed circles.

25. Show that the area of $ABC = 4Rr \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C$.

26. Circles passing through the centre of the circumscribed circle, and having a, b, c as chords, have for radii k, l, m). Prove that $(a \div k) + (b \div l) + (c \div m) = 8 \sin A \sin B \sin C$.

27. Show how to determine the height of a cloud from its angle of elevation and the angle of depression of its reflection in a lake h feet below the level of the observer.

28. A man sees two spires, the tops of which seem to be in a straight line whose angular elevation is m; the angles of depression of the reflections in a lake h feet below him are B and C. Determine the distance between the spires, and the height of each.

29. In any triangle, $R =$
$\frac{1}{4}(a \cot A \csc B \csc C + b \cot B \csc A \csc C + c \cot C \csc B \csc A)$.

30. Determine the radii (r', r'', r''') of the escribed circles, in terms of the sides. Hence show that

$$\frac{1}{r'}+\frac{1}{r''}+\frac{1}{r'''}=\frac{s}{\text{area}} \quad (1); \quad \text{area}=\sqrt{r r' r'' r'''} \quad (2);$$

$$r'+r''+r'''-r=4R \quad (3).$$

31. A, walking along a straight road, observes that the greatest angle that two distant spires subtend at his eye is α; walking c rods farther he observes that they are in a line making an angle β with the road. Find the distance between the spires.

32. A spire S is observed from A and B. If the angles SAB and SBA, the distance AB, and the angle of elevation of the spire at B be known, find the distance of the spire from A and its height above the level road AB.

33. The angular elevation of a hill, as observed from a point due north of it, is A; as observed from a point due west of the first and m miles distant across a lake, the angle of elevation is B. How high is the hill?

34. An observer at sea takes the angles subtended at his position, (P), by the known shore distances, AB, AD, DB. Find AP, DP, BP, when $APD = 42° 17'$, $BPD = 36° 25'$, $AD = 2100$, $BD = 1672$, $AB = 2867$.

35. To determine the geocentric distance of a heavenly body M, knowing the zenith distances of M at two stations of which the latitudes are known, and which are situated on the same meridian.

36. To determine the area of the greatest pentagonal star that can be cut from a circular disc, four inches in diameter.

37. The area of the base of a regular quadrangular pyramid is 289 square feet. Find the area of a section inclined at an angle of $26° 37'$ to the base, one of the sides of the section being parallel to the base, and at a distance therefrom equal to two-thirds of the altitude.

38. The sides of an inscribed quadrilateral are a, b, c, d. What is the radius of the circumscribing circle?

39. If a triangle be inscribed in ABC, its perimeter must lie between $a+b+c$ and $a\cos A+b\cos B+c\cos C$, $(=4R\sin A\sin B\sin C)$.

CHAPTER V.

1. Solve the spherical triangle ABC, given:—

(1) $a=106°\,17'\,22''$; $b=120°\,08'\,20''$; $A=39°\,52'$;

(2) $A=129°\,05'\,28''$; $B=140°\,25'$; $C=110°\,27'$;

(3) $a=70°\,32'$; $b=38°\,25'$; $C=51°\,17'\,24''$;

(4) $A=49°\,37'$; $B=126°$; $b=110°\,17'$;

(5) $C=50°\,32'$; $B=136°\,15'\,28''$; $a=68°\,35'$.

2. To solve a spherical triangle, given

c, B, and $A+C$, or c, B, and $a+c$.

3. Establish the propositions concerning spherical triangles usually given in the text-books in Geometry, by means of the formulæ of chapter V.

4. Prove directly from a figure that

$$\sin a:\sin c::\sin A:\sin C.$$

5. Show that the values given (47) for $\sin\frac{1}{2}a$, $\tan\frac{1}{2}a$, etc., are real.

6. Show that $\sin a:\sin A::$

$$\sin b\sin c:\sqrt{1-\cos^2 a-\cos^2 b-\cos^2 c+2\cos a\cos b\cos c}.$$

7. Show that the sum of any two angles of a spherical triangle is less than $180°+$ the third. What property of a right triangle is suggested hereby?

8. If OX, OY, OZ, be mutually at right angles, and any two lines be drawn through O,

(1) The sum of the squares of the cosines of the angles that each makes with OX, OY, and OZ, respectively, will equal unity.

(2) The cosine of the angle between the two lines will equal the sum of the products of the cosines of the angles made by the two lines, with OX, with OY, and with OZ, respectively.

9. If CD be a median in ABC,

$$\cos AC + \cos CB = 2 \cos \tfrac{1}{2}AB \cos CD.$$

10. If two angles of a spherical triangle be equal to the opposite sides respectively, the remaining angle will be equal or supplementary to the third side.

11. D is any point in AB; show that

$$\cos AD \sin BC = \cos AB \sin DC + \cos AC \sin DB.$$

12. If arcs be drawn from the vertices of any spherical triangle through any point, the product of the sines of one set of alternate segments of the sides will equal the product of the sines of the other set; and conversely.

13. The last proposition proves the concurrence of what sets of angle-transversals of a spherical triangle?

14. If an arc cuts the sides of ABC, in D, E, and F, respectively, what relation obtains between the two sets of alternate segments?

15. If k, l, m, be the altitudes of a spherical triangle,

$$\sin k \sin a = \sin l \sin b = \sin m \sin c;$$

also,
$$\cos m = \csc c \sqrt{\cos^2 a + \cos^2 b - 2 \cos a \cos b \cos c}.$$

16. San Francisco is 122° 25′ 40.″76 W., 37° 47′ 22.″55 N., and Yokohama is 139° 40′ 27″ E., 35° 26′ 50″ N.; what is the shortest sailing distance between the cities? What is the shortest distance on the surface of the earth between Mount Hamilton and Annapolis, Md.? (See Ex. 22).

17. Show that in any right spherical triangle (A = 90°).

$$\sin^2 \tfrac{1}{2}a = \sin^2 \tfrac{1}{2}b \cos^2 \tfrac{1}{2}c + \cos^2 \tfrac{1}{2}b \sin^2 \tfrac{1}{2}c.$$

18. If α, β, γ, be the lengths of the angle-bisectrices of ABC,

$$\cot \alpha \cos \tfrac{1}{2}A + \cot \beta \cos \tfrac{1}{2}B + \cot \gamma \cos \tfrac{1}{2}C = \cot a + \cot b + \cot c.$$

19. Determine the arcual radii (polar distances), of the circumscribed, inscribed, and escribed circles, of the triangles formed by the intersection of any three arcs (of great circles), on the sphere.

20. ABC is equilateral, P the pole of its circumscribing circle, and Q any point on the sphere. To show that

$$\cos QA + \cos QB + \cos QC = 3 \cos PA \cos PQ.$$

What is the analogous formula for the plane triangle?

21. The angle of elevation of the top of a cliff, A, is 23° 27′ 12″; the angle of elevation of a cliff, B, is 10° 28′ 53″; the angle subtended at the point of observation by AB is 73° 47′ 18″. What is the horizontal angle between A and B?*

22. ABC is trirectangular; P is the pole of its circumscribed circle, and Q any point on the sphere. Show that

$$(\cos AQ + \cos BQ + \cos CQ)^2 = 3 \cos^2 PQ.$$

23. In any triangle, $\frac{1}{2}(a+c)$ and $\frac{1}{2}(A+C)$ agree in species.

24. If D, E, F, be the feet of the altitudes of ABC,

$$\tan BD \cdot \tan CE \cdot \tan AF = \tan DC \cdot \tan EA \cdot \tan FB.$$

25. Through any point, in the side of a spherical triangle, to draw an arc (of a great circle), that shall cut off a given fraction of the area.

26. Three circles are described in a triangle, each angle of which is 120°, so that each circle touches the two others and two sides of the triangle. Find the arcual radii.

27. To find the angle formed by two adjacent faces of a regular polyedron.

28. To find, for any regular polyedron, the radii of the inscribed and circumscribed spheres.

29. To find, for *any* tetraedron the radii of the circumscribed, the inscribed, and the escribed, spheres.

30. To express the volume of any tetraedron, (and hence of any parallelopiped), in terms of the edges.

31. If, with the vertices of any parallelopiped as centres, equal spheres be described, the sum of the intercepted portions of the parallelopiped will equal the volume of any one of the spheres.

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* The solution of examples of this character is called "reducing angles to the horizon."

CHAPTER VI.

1. Show that :—

(1) $\cosh^2 x - \cosh^2 y = \sinh^2 x - \sinh^2 y = \sinh(x+y)\sinh(x-y)$

(2) $\cosh^2 y + \sinh^2 x = \cosh^2 x + \sinh^2 y = \cosh(x+y)\cosh(x-y)$;

(3) $\tanh\frac{x}{2} = \frac{\sinh x}{1+\cosh x} = \coth x - \operatorname{csch} x = \coth\frac{x}{2} - 2\operatorname{csch} x$;

(4) $\cosh(x+y) = \frac{1+\tanh x \tanh y}{\operatorname{sech} x \operatorname{sech} y}$;

(5) $\cosh(n+2)k = 2\cosh k \cosh(n+1)k - \cosh nk$.

What are the corresponding formulæ for circular functions?

2. Prove that

$$\tan^{-1}x\sqrt{\frac{bp-aq}{a(p+qx^2)}} = \cos^{-1}\sqrt{\frac{a(p+qx^2)}{p(a+bx^2)}}.$$

What is the corresponding formulæ for inverse hyperbolic functions?

3. Show that

$$\ln\frac{\sqrt{(a+b)\cos\frac{1}{2}x}+\sqrt{(a-b)\sin\frac{1}{2}x}}{\sqrt{(a+b)\cos\frac{1}{2}x}-\sqrt{(a-b)\sin\frac{1}{2}x}} = 2\tanh^{-1}\left\{\frac{\sqrt{a-b}}{\sqrt{a+b}}\tan\tfrac{1}{2}x\right\}.$$

Find corresponding logarithmic equivalents for

$$\tan^{-1}\left\{\frac{\sqrt{a-b}}{\sqrt{a+b}}\tan\tfrac{1}{2}x\right\}, \text{ and } \tanh^{-1}\left\{\frac{\sqrt{a-b}}{\sqrt{a+b}}\tanh\tfrac{1}{2}x\right\}.$$

4. Show that :—

(1) $\cosh(\ln y) + \sinh(\ln y) = y$;

(2) $\tanh(\ln y) = (y^2-1)\div(y^2+1)$. Thence show that

$$\ln 2 = 6.93147\,18+.$$

(3) $\cosh\left(\ln\sqrt{\frac{1+y}{1-y}}\right) = \frac{1}{\sqrt{1-y^2}}$; $\tanh\left(\ln\sqrt{\frac{1+y}{1-y}}\right) = y$.

(4) If $y^2 = 2w-1$, $\tanh(\ln y) = 1 - w^{-1}$.

(5) $\cosh^{-1}(\sec x) = \tanh^{-1}(\sin x) = 2\tanh^{-1}(\tan\frac{1}{2}x)$

$$= \ln\frac{1+\tan\frac{1}{2}x}{1-\tan\frac{1}{2}x} = \ln\tan\tfrac{1}{2}\left(\frac{\pi}{2}+x\right).$$

(6) If $\tan \frac{1}{2} x = z$,

$$\sinh x = \frac{2z}{1-z^2}; \quad \cosh x = \frac{1+z^2}{1-z^2}.$$

(7) $$\cosh^{-1} \frac{ax+b}{\sqrt{b^2-ac}} = \sinh^{-1} \sqrt{\frac{a^2x^2+2abx+ac}{b^2-ac}}.$$

(8) $$\ln \frac{a+x}{a-x} \qquad \tan^{-1} \frac{x}{a}.$$

(9) $$\ln \frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1} = \tanh^{-1} \frac{x\sqrt{2}}{1+x^2}.$$

5. Find the hyperbolic functions of $\frac{1}{4}\pi i$, $\frac{1}{6}\pi i$, and $\frac{1}{12}\pi i$.

6. Calculate directly sinh 1 and cosh 1; thence show that

$$e = 2.718281828459045235 36+.$$

Show also that

$$\sin 1 = 0.841470984807896+;$$
$$\cos 1 = 0.540302305868039+.$$

7. Apply De Moivre's theorem to hyperbolic functions so as to deduce, for the powers of sinh k and cosh k formulæ analogous to those given, for circular functions, on pages 4, 5.

8. Show that:—

$$\ln \sinh k = k - \ln 2 - e^{-2k} - \tfrac{1}{2}e^{-4k} - \tfrac{1}{3}e^{-6k} - \tfrac{1}{4}e^{-8k} - \ldots;$$
$$\ln \cosh k = k - \ln 2 + e^{-2k} - \tfrac{1}{2}e^{-4k} + \tfrac{1}{3}e^{-6k} - \tfrac{1}{4}e^{-8k} + \ldots.$$

Thence derive the development of ln tanh k.

9. Show that

$$\cosh(x+k) = \cosh k + \sinh k \frac{x}{1} + \cosh k \frac{x^2}{2!} + \sinh k \frac{x^3}{3!} + \ldots;$$

$$\sinh(x+k) = \sinh k + \cosh k \frac{x}{1} + \sinh k \frac{x^2}{1!} + \cosh k \frac{x^3}{3!} + \ldots.$$

What are the corresponding developments for the circular functions?

10. Prove the truth of the following equations:—

$$\cosh 2x = 2\cosh^2 x - 1,$$
$$\cosh 3x = 4\cosh^3 x - 3\cosh x,$$
$$\cosh 4x = 8\cosh^4 x - 8\cosh^2 x + 1,$$
$$\cosh 5x = 16\cosh^5 x - 20\cosh^3 x + 5\cosh x,$$

$$\cosh 6x = 32\cosh^6 x - 48\cosh^4 x + 18\cosh^2 x - 1,$$
$$\cosh 7x = 64\cosh^7 x - 112\cosh^5 x + 56\cosh^3 x - 7\cosh x,$$
$$\cosh 8x = 128\cosh^8 x - 256\cosh^6 x + 160\cosh^4 x - 32\cosh^2 x + 1.$$

Establish the corresponding formulæ for the hyperbolic sine, and tangents. Express also $\sinh^2 x$, $\sinh^3 x$, $\cosh^2 x$, $\cosh^3 x$, etc., in terms of functions of x. Compare Ex. **4**, page 5.

11. If $n = 2r$, $x = a + b\cot\theta$,

$$\frac{(A + iB)(x - a + bi)^n + (A - iB)(x - a - bi)^n}{\{(x-a)^2 + b^2\}^n} = \frac{A\cos n\theta - B\sin n\theta}{\{(x-a)^2 + b^2\}^r}.$$

12. Show that, as

$$\ln(\cos k + i\sin k) = ki,$$

the logarithm of unity has an infinite number of values, of which all but one are imaginary. Thence show that any number has an infinite number of logarithms for any given base.

13. Show that

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots\ldots$$

This may be shown by proving, first, that the equation $\sin\theta = 0$ can have no imaginary roots, and then showing that the series (see page 49) for $\sin\theta$ is exactly divisible by $\theta \pm n\pi$, n being any integer. Thus $\sin\theta$ is shown to be a product of the form

$$a\,\theta(\theta+\pi)(\theta-\pi)(\theta+2\pi)(\theta-2\pi)(\theta+3\pi)(\theta-3\pi)\ldots\ldots$$

in which a cannot contain θ, since the equation $\sin\theta = 0$ has no imaginary roots. Finally it may be shown that $a = 1$, whence we may factor $\sin\theta$ as shown. $\cos\theta$ may, of course, be treated similarly.

From these expressions for $\sin\theta$ and $\cos\theta$ we may deduce interesting forms of algebraic statement for the other exponential functions.

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